# Integrability in two-dimensional gravity

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# Abstract

In this thesis, we study gravity and supergravity systems that become completely integrable in two dimensions. Including Einstein gravity, these systems are theories that upon dimensional reduction to three dimensions assume the form of a non-linear $\sigma$-model for the matter part, with target manifold a coset space G/K. Starting from Einstein gravity and focusing on the class of stationary axisymmetric solutions, we study the linear system (Lax pair) associated with the non-linear field equations of vacuum gravity as formulated by Belinski - Zakharov (BZ) and Breitenlohner-Maison (BM). The existence of the linear system exhibits the integrability of the two-dimensional system and is amenable to inverse scattering methods as shown in two different approaches by BZ and BM. The infinite dimensional symmetry associated with the two-dimensional equations gives rise to the so-called Geroch group. The BM approach allows for a direct implementation of the Geroch group and the generation of physically interesting solutions in the soliton sector in a manifestly group theoretic way. For this reason, it is expected to apply to a broader set of coset models. Throughout this work, we concentrate on this approach and extend it to STU supergravity, where appropriate technical modifications were required in the BM solution generation algorithm. Based on these modifications, we also discuss a generalization to other set-ups. We test the applicability of the BM inverse scattering method by explicitly constructing the Kerr-NUT solution of Einstein gravity and within STU supergravity, the four-charge black hole solution of Cvetic and Youm as well as the singly rotating JMaRT solution.

**Keywords:**
Two-dimensional gravity, integrability, STU supergravity, solution generation methods, gravitational solitons

# Zusammenfassung

In dieser Arbeit untersuchen wir Gravitations- und Supergravitationssysteme, die in zwei Dimensionen vollständig integrabel sind. Dies sind Theorien, zu denen auch die einsteinsche Gravitation zählt, die bei dimensionaler Reduktion auf drei Dimensionen, die Form eines nichtlinearen $\sigma$-Models für den Materieteil annehmen und als Zielmannigfaltigkeit den Cosetraum G/K haben. Ausgehend von der einsteinschen Gravitation betrachten wir insbesondere die Klasse der stationären und axialsymmetrischen Lösungen. Dabei untersuchen wir das lineare System (Lax-Paar), das den nichtlinearen Feldgleichungen der Vakuumsgravitation entspricht, wie es von Belinski-Zakharov (BZ) und Breitenlohner-Maison (BM) formuliert wurde. Die Existenz des linearen Systems zeigt die Integrabilität des zweidimensionalen Systems und ist inversen Streumethoden zugänglich, wie in zwei unterschiedlichen Ansätzen von BZ und BM gezeigt. Aus der unendlich-dimensionalen Symmetrie, die mit den zweidimensionalen Gleichungen assoziiert ist, ergibt sich die sogenannte Gerochgruppe. Der BM-Ansatz ermöglicht eine direkte Implementierung der Gerochgruppe und der Erzeugung von physikalisch interessanten Lösungen im Solitonensektor auf manifest gruppentheoretischer Weise. Aus diesem Grund ist zu erwarten, dass es in einem breiteren Spektrum von Cosetmodellen angewendet werden kann. In dieser Arbeit konzentrieren wir uns auf diesen Ansatz und erweitern ihn um die STU-Supergravitation, wobei entsprechende technische Änderungen im BM-Lösungserzeugungsalgorithmus erforderlich werden. Basierend auf diesen Änderungen, diskutieren wir auch eine Verallgemeinerung auf andere Fälle. Wir testen die Anwendbarkeit der BM inversen Streumethode, indem wir explizit folgende Lösungen konstruieren: die Kerr-NUT Lösung der einsteinschen Gravitation, die Vier-Ladungs-Lösung eines schwarzen Lochs innerhalb der STU Supergravitation von Cvetic und Youm und die einfach rotierende JMaRT Lösung.

**Schlagwörter:**
Zweidimensionale Gravitation, Integrabilität, STU-Supergravitation, Lösungserzeugungsmethoden, gravitational solitons

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# Chapter 1

# Introduction

Since the advent of general relativity, the search for exact solutions of Einstein's field equations has attracted a lot of interest. Considerable effort has been put into finding techniques to systematically construct such solutions in order to study their properties and thus deepen our understanding of gravity. The main barrier in the course of this noble endeavour is the complicated nature of the Einstein field equations. It is a set of coupled, non-linear partial differential equations whose solution poses a very challenging if not insurmountable problem.

However, under special circumstances, as in the case of spacetimes with enough symmetries, (e.g. stationarity and axial symmetry) the Einstein equations simplify significantly. Even though it remains a challenging problem, there are methods that can be employed to treat it. Among the authors who worked on the development of solution generation methods were Ehlers [1], Geroch [2, 3], Hoenselaers, Kinnersley and Xanthopoulos [4], Harrison [5], Hauser and Ernst [6], Cosgrove [7, 8], Belinski and Zakharov [9, 10, 11]. Their work and that of many others (the literature exploring this topic is too extensive for an exhaustive account) contributed significantly to the subject of exact solutions in general relativity. (For a general reference documenting exact solutions, see [12]).

The main observation, even before solution generation methods were developed, is that (effectively) two-dimensional gravity is a completely integrable system. The symmetry underlying the system of equations is infinite dimensional, giving rise to infinitely many conserved quantities. This observation was first made by Geroch in [3] for the class of stationary axisymmetric solutions of vacuum Einstein gravity. Geroch showed that each solution involves infinitely many potentials which in turn give rise to an infinite parameter set of transformations acting on this solution. The generation of new solutions from initial seed solutions through this set of transformations is referred to as the Geroch symmetry of two-dimensional gravity.

The integrability of the two-dimensional system is exhibited by the existence of a "Lax pair", i.e. a system of linear equations that is equivalent to the problem of interest. The first ones to show that a linear system can be written for the (two-dimensional) Einstein field equations were Belinski and Zakharov (BZ) [9, 10].

Moreover, they managed to adapt the inverse scattering method used in other non-linear integrable problems (Korteweg-de Vries, Sine-Gordon, non-linear Schrödinger equation) to construct solitonic solutions of the gravitational equations. Among these "gravitational solitons" are many physically interesting solutions such as black holes, colliding plane waves and cosmological solutions [11].

Subsequent studies by Breitenlohner and Maison [13, 14], based on work by Geroch [3] and Julia [15, 16], revealed a better understanding of the group theoretic structure of reduced gravity and provided a formulation of the theory and the linear system that is suited to this picture. In this research direction, much has been contributed by the authors in [17, 18, 19, 20, 21, 22, 23, 24].

For a class of theories [14] including Einstein gravity and other supergravity theories, reduction to three dimensions (due to presence of Killing isometries) results in a gravity-matter system, with the structure of a non-linear $\sigma$-model for the matter part. This means that apart from the pure gravity part, the theory involves only scalar fields with values in a target manifold. The common characteristic of gravity theories that attain this form is that the target space is a Cartan symmetric space G/K, where G is the group of global Ehlers symmetry transformations and $K$ is a local symmetry group that is maximal compact subgroup of $G$ (or a subgroup of equal dimension when G/K is a pseudo-Riemannian symmetric space).

Moreover, the Geroch transformations of [2, 3] fit into this picture, when the theory is further reduced to two dimensions. Then, the symmetries become greatly enhanced and the transformations in the space of solutions constitute the so-called Geroch group. The latter is in fact an infinite dimensional group whose associated Lie algebra is a Kac-Moody algebra [25, 13, 15]. What is more, there is a practical side to the study of this structure, that has led to a group theoretic view on solution generation. Notably, the Geroch symmetry acts transitively on a given class of solutions and is large enough to contruct all solutions in this class from the simplest one, (e.g. Minkowski space in D=4 vacuum gravity) [6, 13, 22]. In [13, 26], Breitenlohner and Maison (BM) describe the linear system and an algorithm to generate solitonic solutions based on the action of the Geroch group on known seed solutions. In four- and five- dimensional gravity, albeit illuminating from the group theory point of view, the method of Breitenlohner-Maison is not as efficient and practical as that of Belinski and Zakharov. However, since it is a manifestly group theoretic method that is not strongly tailored on certain groups, it has the potential of being generalized to other settings. This could provide a systematic way to generate solutions of theories beyond Einstein gravity, like supergravity models in the G/K class mentioned earlier.

In this dissertation, we follow the work of Breitenlohner and Maison [13, 26] starting from Einstein gravity in four dimensions and subsequently, through general technical adjustments, extending it to theories with different symmetry groups. The concrete application beyond Einstein gravity is on the STU supergravity model, within which we construct explicit examples of (known) solutions generated via the BM method.

The structure of this thesis is as follows. The next chapter is an introduction on the topic of infinite-dimensional symmetries in reduced gravity, following the simplest case of Einstein gravity for concreteness. In the third chapter, we present the solution generation methods based on inverse scattering by Belinski-Zakharov (BZ) as well as Breitenlohner-Maison (BM) and discuss their interrelations. We close this chapter with an explicit example, the Kerr-NUT solution, not previously generated via the BM method[1]. Next, in chapter 4, we study the generalisation of the BM method that accommodates other (Ehlers) symmetry groups and specifically adjust it to the four-dimensional STU supergravity model. The coset space associated to the reduction in three dimensions is G/K $=\mathrm{SO}(4,4)/(\mathrm{SO}(2,2)\times\mathrm{SO}(2,2))$ and using our generalized version of the BM technique we construct an explicit example, namely the four-charge black hole of Cvetic and Youm [28]. Following that, in chapter 5, we take the method a step further such that it allows for generation of solutions which are asymptotically flat in five dimensions. This allows for the construction of five-dimensional black objects such as the Myers-Perry solution [29]. With the Myers-Perry instanton as a starting point, we are able to reach the singly-rotating JMaRT solution [30] as an uplift to six dimensions. Finally, chapter 6 consists of concluding remarks as well as future directions and open problems in this research topic.

[1] The case for Schwarzschild and Kerr solution is presented in [26, 27].

# Chapter 2

# Symmetries in dimensionally reduced gravity

In this chapter we will set the scene for the following ones in terms of structure of the theories we study as well as notation. Since the general set-up and characteristics of this class of gravity systems are similar, we choose to present the case of Einstein gravity for more simplicity and clarity. This case is very well documented in the literature (e.g. [13], [27], [31], [32]) and the aim of reviewing it here is to provide an introductory basis of our topic. We will discuss its symmetry properties when it is reduced to an effectively three- and two-dimensional theory in the presence of Killing isometries. Performing the reduction in a "Kaluza-Klein" way, we will arrive at a gravity-matter system with the structure of a non-linear $\sigma$-model connected to a certain symmetry group. In the further reduced two-dimensional theory, the equations of motion are characterized by an infinite dimensional symmetry, called the Geroch symmetry. This implies the complete integrability of the theory that is exhibited by means of a linear system of equations (Lax pair) that are amenable to solution generation techniques, such as the inverse scattering transform, explained in subsequent chapters.

## 2.1 Reduction to two dimensions

Let us start with the Einstein-Hilbert action

$$S = \int d^4x \sqrt{-g}R, \tag{2.1}$$

where $g$ is the determinant of the four dimensional metric $g_{MN} = E_M{}^A E_N{}^B \eta_{AB}$, $E_M{}^A$ the vierbein and $\eta_{AB} = (-+++)$. We will focus on spacetimes which possess two commuting orthogonal Killing vector fields such that the equations of motion retain dependence on two variables only. For concreteness let us consider stationary, axisymmetric spacetimes, that is spacetimes with a timelike and a spacelike Killing vector. Choosing a coordinate system which uses the Killing parameters as in $(x^0 = t, x^1 = \phi, x^2, x^3)$, the Killing vectors are the coordinate vector fields $\partial_t, \partial_\phi$. The metric components will then depend only on the coordinates $x^2, x^3$. At the level of the Lagrangian, the presence of the isometries corresponds to an effective dimensional reduction of the theory. In the following we will outline this dimensional reduction as performed in the spirit of the Kaluza-Klein programme [32]. Starting from (2.1), we will carry out the reduction in two steps, namely by reducing to three and then to two dimensions. In the step from three to two dimensions, we can reach the final two-dimensional theory in two ways. As will soon become clear, studying both these processes as well as their interrelations will serve to illustrate the symmetry properties of the reduced gravity theory.

### 2.1.1 The *Ehlers* Lagrangian

**From four to three dimensions**

Starting from four dimensions and reducing along the timelike Killing direction first, we use the ansatz for the vierbein:

$$E_M{}^A = \begin{pmatrix} \Delta^{-1/2} e_m{}^a & \Delta^{1/2} B_m \\ 0 & \Delta^{1/2} \end{pmatrix}, \qquad E_A{}^M = \begin{pmatrix} \Delta^{1/2} e_a{}^m & -\Delta^{1/2} e_a{}^m B_m \\ 0 & \Delta^{-1/2} \end{pmatrix} \tag{2.2}$$

and thus the metric is written as

$$g_{MN} = \begin{pmatrix} \Delta^{-1} g_{mn} - \Delta B_m B_n & -\Delta B_m \\ -\Delta B_m & -\Delta \end{pmatrix} \tag{2.3}$$

$$ds^2 = -\Delta (dt + B_m dx^m)^2 + \Delta^{-1} g_{mn} dx^m dx^n. \tag{2.4}$$

The three-bein for the orbit space of the action of the Killing field is $e_m{}^a$ and $B_m, \Delta$ are the Kaluza-Klein vector and scalar respectively. The capital indices correspond to four-dimensional quantities and latin lower case indices to three-dimensional ones. Moreover, (2.2) is brought to triangular form by use of the local Lorentz invariance. After some calculations, the ansatz (2.2) leads to a Lagrangian written in terms of three-dimensional quantities :

$$\mathcal{L}^{(3d)} = \sqrt{g_3} \left( R^{(3d)} - \frac{1}{2} g^{mn} \Delta^{-2} \partial_m \Delta \partial_n \Delta + \frac{1}{4} \Delta^2 B^{mn} B_{mn} \right), \tag{2.5}$$

where $\sqrt{g_3} = \Delta \sqrt{-g}$ , $g_{mn} = e_m{}^a e_n{}^b \eta_{ab}$ with $\eta_{ab} = (+++)$ the three-dimensional metric on the orbit space and $B_{mn} = \partial_m B_n - \partial_n B_m$.

The matter terms in the above Lagrangian can become purely scalar, by dualizing the three-dimensional Kaluza-Klein vector field $B_m$ into a scalar field. To achieve this, we treat $B_{mn}$ as an independent field in the Lagrangian and impose its Bianchi identity, using the following observation. A term of the form

$$\mathcal{L}' = \frac{1}{2} \sqrt{g_3} \varepsilon^{mnk} B_{mn} \partial_k \tilde{\psi}, \tag{2.6}$$

can be safely added to (2.5), since it can be dropped by integration by parts and using the Bianchi identity for the tensor $B^{mn}$. The scalar field $\tilde{\psi}$ is introduced for now as a Lagrange multiplier. If we vary the new Lagrangian $\mathcal{L}'' = (\mathcal{L} + \mathcal{L}')$ with respect to $B_{mn}$ we find the equation

$$B^{mn} = \frac{1}{\sqrt{g_3}} \Delta^{-2} \epsilon^{mnk} \partial_k \tilde{\psi}, \tag{2.7}$$

which we use to substitute for $B_{mn}$ in $\mathcal{L}''$ ($\epsilon^{mnk} = \frac{1}{\sqrt{g_3}} \varepsilon^{mnk}$ with $\varepsilon_{mnk}$ the totally antisymmetric Levi-Civita tensor density). The resulting Lagrangian reads

$$\mathcal{L}^{(3d)} = \sqrt{g_3} \left( R^{(3d)} - \frac{1}{2} g^{mn} \Delta^{-2} \left( \partial_m \Delta \partial_n \Delta + \partial_m \tilde{\psi} \partial_n \tilde{\psi} \right) \right) \tag{2.8}$$

and has the form of pure gravity in three dimensions coupled to a scalar matter part. At this stage, we are able to show that (2.8) enjoys a global $\mathrm{SL}(2,\mathbb{R})$ symmetry: using the complex field $\mathcal{T} = \tilde{\psi} + i\Delta$ we can write

$$\mathcal{L}^{(3d)} = \sqrt{g_3} R^{(3d)} - 2\sqrt{g_3} g^{mn} \frac{\partial_m \mathcal{T} \partial_n \bar{\mathcal{T}}}{(\mathcal{T} - \bar{\mathcal{T}})^2} \tag{2.9}$$

and easily check that it is invariant under the transformations

$$\mathcal{T} \to \mathcal{T}' = \frac{a\mathcal{T} + b}{c\mathcal{T} + d} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \,. \tag{2.10}$$

As it turns out, the emergence of a global symmetry in the reduction to three dimensions is a shared feature of a class of gravity theories including and beyond Einstein gravity [16, 14]. We will henceforth refer to this type of symmetry as "Ehlers symmetry" , as it was J. Ehlers who first studied these symmetry properties in the context of Einstein gravity during the 1950's [1].

**From three to two dimensions**

The reduction to two dimensions proceeds with the assumption of an additional (spacelike) Killing field $\partial_\phi$. We start from the three-dimensional theory (2.8) and take the Kaluza-Klein ansatz for the three-bein $e_m{}^a$

$$e_m{}^a = \begin{pmatrix} f_E \, e_\mu{}^\alpha & \rho A_\mu \\ 0 & \rho \end{pmatrix}, \quad e_a{}^m = \begin{pmatrix} f_E^{-1} e_\alpha{}^\mu & -f_E^{-1} e_\alpha{}^\lambda A_\lambda \\ 0 & \rho^{-1} \end{pmatrix}, \tag{2.11}$$

with $e_\mu{}^\alpha$ the two-bein, $A_\mu$ the Kaluza-Klein vector, $\rho$ the Kaluza-Klein scalar (the "dilaton") and $f_E$ is a conformal factor. One simplification that we can employ now is to set the vector field $A_\mu$ to zero, since it carries no propagating degrees of freedom in two dimensions (in the absence of topological reasons that would result in $A_\mu$ developing a non-zero holonomy). After some calculations, we arrive at the Lagrangian

$$\mathcal{L}_E^{(2d)} = \sqrt{g_2}\rho \left( R^{(2d)} + 2g^{\mu\nu} f_E^{-1} \partial_\mu f_E \rho^{-1} \partial_\nu \rho - \frac{1}{2} g^{\mu\nu} \Delta^{-2} \left( \partial_\mu \Delta \partial_\nu \Delta + \partial_\mu \tilde{\psi} \partial_\nu \tilde{\psi} \right) \right), \tag{2.12}$$

where we have used the subscript "$E$" in $\mathcal{L}_E$ to denote the "Ehlers" Lagrangian (in two dimensions) resulting from the above process of reduction.

### 2.1.2 The *Matzner-Misner* Lagrangian

In this section, we will present another path of reduction which leads to an equivalent theory in two dimensions. The resulting Lagrangian is a system first studied by Matzner-Misner in [33] and follows from dimensional reduction from four to two dimensions without dualisation of the Kaluza-Klein vector. Instead, we will split the vector $B_m$ in (2.2) into a vector $B_\mu$ in two dimensions and a scalar $\psi$ as (we write the components in the order $((B_{x^3}, B_{x^2}), B_{x^1})$ :

$$B_m = (B_\mu, \psi) \tag{2.13}$$

which leaves us, in two dimensions, with the metric $g_{\mu\nu}$, two vectors $(B_\mu, A_\mu)$ and two scalars $(\Delta, \psi)$. The vectors can be set to zero, using the same argument as before, namely that Maxwell fields carry no propagating degrees of freedom in two dimensions[1]. Therefore, starting from (2.5) and using (2.13) we have that

$$g^{mp} g^{nq} B_{mn} B_{pq} = 2 f_E^{-2} \rho^{-2} g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi \tag{2.15}$$

and with ($\sqrt{g_3} = f_E^2 \rho \sqrt{g_2}$) we get

$$\begin{aligned}\mathcal{L}^{(2d)} =& \sqrt{g_2}\rho\Big( R^{(2d)} + 2g^{\mu\nu} f_E^{-1} \partial_\mu f_E \rho^{-1} \partial_\nu \rho - \frac{1}{2} g^{\mu\nu} \Delta^{-2} \partial_\mu \Delta \partial_\nu \Delta + \\ &+ \frac{1}{2} \rho^{-2} \Delta^2 g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi \Big),\end{aligned} \tag{2.16}$$

where $g_{\mu\nu} = e_\mu{}^\alpha e_\nu{}^\beta \eta_{\alpha\beta}$. It is useful for discussions that follow to write (2.16) as

$$\begin{aligned}\mathcal{L}_{MM}^{(2d)} =& \sqrt{g_2}\rho\Big( R^{(2d)} + 2g^{\mu\nu} f_{MM}^{-1} \partial_\mu f_{MM} \rho^{-1} \partial_\nu \rho - \frac{1}{2} g^{\mu\nu} \hat{\Delta}^{-2} \big( \partial_\mu \hat{\Delta} \partial_\nu \hat{\Delta} \\ &- \partial_\mu \psi \partial_\nu \psi \big) \Big),\end{aligned} \tag{2.17}$$

[1] Another way to arrive at the same conclusion is through the hypersurface orthogonality of the Killing fields. Indeed, for spacetimes with two commuting Killing fields, satisfying hypersurface orthogonality conditions (see e.g. ch.7 in [34]), the metric can be written in block diagonal form as

$$g_{MN} = \begin{pmatrix} h_{\mu\nu} & 0 \\ 0 & \bar{h}_{\bar{\mu}\bar{\nu}} \end{pmatrix}, \tag{2.14}$$

with $h_{\mu\nu}$ the metric on the orbit space (with coordinates $(x^2, x^3)$) and $\bar{h}_{\bar{\mu}\bar{\nu}}$ the internal metric on the surface with coordinates $(t, \phi)$.

where we have set
$$\hat{\Delta} = \frac{\rho}{\Delta}, \qquad f_{MM} = f_E \rho^{1/4} \Delta^{-1/2} \tag{2.18}$$
and the subscript "MM" stands for Matzner-Misner. To bring the Lagrangian in the form (2.17), we start from (2.16) where we add and subtract the term $(-\frac{1}{2}\rho^{-1}\partial_\mu \rho \partial^\mu \rho + \Delta^{-1}\partial_\mu \rho \partial^\mu \Delta)$. With the definitions (2.18), we have that
$$-\frac{1}{2}\rho^{-1}\partial_\mu \rho \partial^\mu \rho + \Delta^{-1}\partial_\mu \rho \partial^\mu \Delta - \frac{1}{2}g^{\mu\nu}\Delta^{-2}\partial_\mu \Delta \partial_\nu \Delta = -\frac{1}{2}g^{\mu\nu}\hat{\Delta}^{-2}\partial_\mu \hat{\Delta}\partial_\nu \hat{\Delta},$$
$$\frac{1}{2}\rho^{-1}\Delta^2 \partial_\mu \psi \partial^\mu \psi = \frac{1}{2}\rho \hat{\Delta}^{-2}\partial_\mu \psi \partial^\mu \psi$$
and
$$2f_E^{-1}\partial_\mu f_E \rho^{-1}\partial^\mu \rho + \frac{1}{2}\rho^{-1}\partial_\mu \rho \partial^\mu \rho - \Delta^{-1}\partial_\mu \rho \partial^\mu \Delta = 2f_{MM}^{-1}\partial_\mu f_{MM}\rho^{-1}\partial^\mu \rho. \tag{2.19}$$

The gravity-matter Lagrangians (2.12) and (2.17) look very similar in form. Note that, apart from the identifications (2.18), the two systems are also connected through the duality relation that is derived from (2.7) after reduction to two dimensions:
$$\rho^{-1}\Delta^2 \partial_\mu \psi = {}^*\partial_\mu \tilde{\psi}, \tag{2.20}$$
where ${}^*\partial_\mu \tilde{\psi} = \epsilon_{\mu\nu}\partial^\nu \tilde{\psi}$.

As we will see shortly, the Ehlers and Matzner-Misner systems can be described in terms of distinct non-linear $\sigma$-models of scalar fields assuming values in a target manifold. In the next section, we will first provide some general features of such $\sigma$-models and then proceed to the specific cases of (2.12) and (2.17) .

## 2.2 Gravity as a non-linear $\sigma$-model

In general, the action of a gravity-matter system with matter in the form of a $\sigma$-model of scalar fields $\phi^i$ reads
$$S = \int_M dx\sqrt{-g}\left(R - \frac{1}{2}g^{mn}\partial_m \phi^i \partial_n \phi^j h_{ij}\right), \tag{2.21}$$
where $M$ is the space-time manifold with metric $g_{mn}$. The scalar fields $\phi^i$ assume values on a non-linear target manifold with metric $h_{ij}$.

In Einstein gravity as well as for a large class of supergravity models, this $\sigma$-model structure appears already in three dimensions and the target manifold is a non-compact Riemannian symmetric space G/K. The group G is the Ehlers group of global symmetry transformations that leave (2.21) invariant and K is the maximal compact subgroup of G (or a subgroup of equal dimension in the case that G/K is a pseudo-Riemannian symmetric space) determined by an involutive automorphism.

To illuminate the group theoretical aspects of such gravity-matter systems, we will rewrite the Lagrangian (2.21), using a suitable parameterization of the coset space G/K in terms of the coordinates $\{\phi^i\}$. To this end, we shall need some additional elements and notation from the study of non-linear $\sigma$-models and symmetric spaces that we will briefly present in what follows.

### Non-linear $\sigma$-models with G/K target manifolds

For the symmetric spaces of interest in this discussion, let us consider a non-compact, real Lie group G and an involutive automorphism $\tau : \mathrm{G} \to \mathrm{G}$, $\tau^2 = \mathrm{id}_{\mathrm{G}}$. There is a subgroup K that is fixed by $\tau$, i.e.

$$\mathrm{K} = \{k \in \mathrm{G} : \tau(k) = k\} \tag{2.22}$$

and the coset space G/K is a non-compact (pseudo-)Riemannian symmetric space. In the case of Riemannian symmetric spaces, K is the maximal compact subgroup of G while for pseudo-Riemannian symmetric spaces the denominator group is no longer compact, but has the same dimension as the maximal compact subgroup.

To parameterize the quotient space G/K, we seek a group element $V(x)$ (parameterized by the $\sigma$-model scalars) to represent each coset. There is no unique choice of such representatives, but a simple choice is to take $V(x)$ to be element of the subgroup of "triangular" matrices in the spirit of Iwasawa decomposition [35] [2]. This constitutes a gauge choice which is not generally preserved under the (global) action of the group G. The transformation rule that preserves the gauge involves a a local transformation $k(x) \in \mathrm{K}$, that generally depends non-linearly on $V(x)$ and $g \in \mathrm{G}$ and reads

$$V(x) \to k(x)V(x)g, \quad g \in \mathrm{G}, \tag{2.24}$$

where $k(x)$ has the role of restoring the triangular form of the transformed element $V$.

[2] According to the Iwasawa decomposition of a Lie algebra, every element in the associated Lie group G (G semisimple) can be written as the product of three elements in a unique way [35] (see also e.g. [36],[37]) :

$$g = g_{\mathrm{K}} g_{\mathrm{H}} g_{\mathrm{N}}, \tag{2.23}$$

where $g_{\mathrm{K}}$ is an element in the maximal compact subgroup K of G, $g_{\mathrm{H}}$ is in the subgroup associated to the Cartan subalgebra of G and $g_{\mathrm{N}}$ is the subgroup arising from the exponentiation of a nilpotent subalgebra. The latter is given by the sum of (restricted) root spaces corresponding to positive roots. (For example, in the case of $\mathrm{G} = \mathrm{SL}(n, \mathbb{R})$, the subgroup K is $\mathrm{SO}(n)$, H is the subgroup of positive diagonal matrices and N is the subgroup of upper triangular matrices with 1 in the diagonal). In light of the above decomposition, the triangular coset representatives $V$ that we discuss here are constructed as $V = g_{\mathrm{H}} g_{\mathrm{N}}$, i.e. by exponentiating the Cartan generators and all the positive-root generators, cf. relations (2.43),(4.32).

It should be noted that for the case where the coset space G/K is a non-compact pseudo-Riemannian symmetric space, the denominator group is non-compact. In this case, the maximal compact subgroup of G is defined by a different involutive automorphism from the one fixing K. The choice of "triangular" coset respresentatives is not globally possible since there are still compact generators remaining in G/K. Because of K being non-compact, the metric $h_{ij}$ in (2.21) will not be positive-definite.

The involution $\tau$ is an automorphism of G and as such induces an automorphism on the Lie algebra $\mathfrak{g}$ of G. The Lie algebra involution, denoted again by $\tau$, squares to the identity, thus splitting the algebra into an invariant and an anti-invariant part

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{2.25}$$

with

$$\tau(\mathfrak{k}) = \mathfrak{k}, \quad \tau(\mathfrak{p}) = -\mathfrak{p}. \tag{2.26}$$

The commutation rules for the subspaces $\mathfrak{k}$,$\mathfrak{p}$ read

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{k}. \tag{2.27}$$

Using the above Lie algebra decomposition, we write the expression $\partial_m V\, V^{-1}$ with values in $\mathfrak{g}$ as

$$\partial_m V\, V^{-1} = P_m + Q_m \tag{2.28}$$

where $P_m$ and $Q_m$ satisfy

$$P_m^\sharp = P_m, \quad Q_m^\sharp = -Q_m \tag{2.29}$$

with $\sharp$ denoting the anti-involution on $X \in \mathfrak{g}$, i.e.

$$X^\sharp = -\tau(X). \tag{2.30}$$

The same notation will be used for the anti-involution on the group elements $g \in$ G, with

$$g^\sharp = \tau(g^{-1}) = \tau(g)^{-1} \quad \text{and} \quad (g_1 g_2)^\sharp = g_2^\sharp g_1^\sharp, \ (g_1, g_2) \in \mathrm{G}. \tag{2.31}$$

The explicit action of the map $\sharp$ on elements in G and Lie(G) depends on the group and will be specified once we consider specific examples. From the transformation (2.24) and the expression (2.28) we infer the transformation laws for $P_m$ and $Q_m$ that read

$$Q_m = \frac{1}{2}\left(\partial_m V V^{-1} - (\partial_m V V^{-1})^\sharp\right) \to k\, Q_m\, k^{-1} + \partial_m k\, k^{-1}, \tag{2.32a}$$

$$P_m = \frac{1}{2}\left(\partial_m V V^{-1} + (\partial_m V V^{-1})^\sharp\right) \to k\, P_m\, k^{-1}, \tag{2.32b}$$

that is $Q_m$ transforms like a gauge field under the action of the local group K while $P_m$ transforms covariantly. Both $Q_m$ and $P_m$ are invariant under the action of the group G.

Using the group involution $\tau$ and the element $V(x)$, we can form a useful object that transforms linearly under G and is invariant under K, namely

$$M = V^\sharp V, \tag{2.33}$$

with transformation

$$M \to g^\sharp\, M\, g, \quad g \in \mathrm{G}. \tag{2.34}$$

Moreover, from (2.28) we have that

$$(\partial_m V - Q_m V)\, V^{-1} \equiv D_m V V^{-1} = P_m \tag{2.35}$$

with $D_m$ the K-covariant derivative. We may relate $P_m$ to $M$ as [3]

$$P_m = D_m V\, V^{-1} = \frac{1}{2} V M^{-1} \partial_m M V^{-1}. \tag{2.36}$$

At this stage, we have introduced all the ingredients to re-write the $\sigma$-model part of (2.21) in terms of group theory objects that enable a better analysis of its symmetry properties. We can define a G-invariant metric $h_{ij}$ on the manifold G/K with coordinates $\phi^i$ as

$$h_{ij} d\phi^i d\phi^j = \langle P, P \rangle, \tag{2.37}$$

where $\langle \cdot, \cdot \rangle$ is an invariant scalar product on $\mathfrak{g}$ that is a positive multiple of the Killing metric when G is a simple group. We will fix this multiplicative factor by setting

$$\langle H_i, H_i \rangle = 4, \tag{2.38}$$

where $\{H_i\}$ are the set of mutually commuting generators in the Chevalley-Serre form of the Lie algebra $\mathfrak{g}$ that generate the Cartan subalgebra of $\mathfrak{g}$. In the following sections and chapters, we will use this representation of the Lie algebra $\mathfrak{g}$ when discussing the $\sigma$-model description of reduced gravity. Using (2.37), the equations of motion derived from (2.21) admit the form

$$R_{mn} - \frac{1}{2} \langle P_m, P_n \rangle = 0 \tag{2.39a}$$

$$D_m(\sqrt{-g} P^m) = 0 \tag{2.39b}$$

with $D_m P^m = \partial_m P^m - [Q_m, P^m]$.

Alternatively, one can express everything in terms of $M$, using (2.36). We can write

$$h_{ij} d\phi^i d\phi^j = \frac{1}{4} \langle M^{-1} dM, M^{-1} dM \rangle \tag{2.40}$$

and then using the above expression we get the field equations in the form

$$R_{mn} - \frac{1}{8} \langle M^{-1} \partial_m M, M^{-1} \partial_n M \rangle = 0 \tag{2.41a}$$

$$D_m(M^{-1} \partial^m M) = 0. \tag{2.41b}$$

Using the language of this section, we will proceed to look into the $\sigma$-model decription of specific examples, namely the systems (2.12) and (2.17).

[3] Let us note the analogy between the elements $(V, M)$ and the vielbein and metric in general relativity. $V$ is an element of the coset space G/K, while the vielbein is also an element of a coset space, namely $\mathrm{GL}(D, \mathbb{R})/\mathrm{SO}(1, D-1)$. Moreover, the "metric" $M$ completely determines the $\sigma$-model and is related to $V$ in a manner analogous to the metric and vielbein.

### 2.2.1 The *Ehlers* $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ $\sigma$-model

In section (2.1.1), we discussed reduction of gravity from four to three dimensions and arrived at the Lagrangian (2.8). Clearly, it has the form of gravity coupled to a $\sigma$-model as in (2.21). Moreover, it is shown to be invariant under the $\mathrm{SL}(2,\mathbb{R})$ transformation (2.10), that we will denote as $\mathrm{SL}(2,\mathbb{R})_E$ from now on, to indicate that it refers to the Ehlers symmetry. The $\sigma$- model metric in (2.8)

$$ds_\sigma^2 = \Delta^{-2}(d\Delta^2 + d\tilde{\psi}^2) \tag{2.42}$$

can be identified as the invariant metric on the space $\mathrm{SL}(2,\mathbb{R})_E/\mathrm{SO}(2)$ with coordinates $(\Delta, \tilde{\psi})$. We choose the triangular coset representative

$$V_E(x) = \mathrm{e}^{-\frac{1}{2}\ln\Delta\, h}\mathrm{e}^{\tilde{\psi}\, e} = \begin{pmatrix} \Delta^{-1/2} & \tilde{\psi}\Delta^{-1/2} \\ 0 & \Delta^{1/2} \end{pmatrix}, \tag{2.43}$$

where $h, e$ refer to the Chevalley-Serre $\mathfrak{sl}(2,\mathbb{R})$ basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{2.44}$$

with commutation relations

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h\,. \tag{2.45}$$

With this choice for $V$, we find the $\mathfrak{sl}(2,\mathbb{R})$ element

$$\partial_m V_E V_E^{-1} = \begin{pmatrix} -\frac{1}{2}\Delta^{-1}\partial_m\Delta & \Delta^{-1}\partial_m\tilde{\psi} \\ 0 & \frac{1}{2}\Delta^{-1}\partial_m\Delta \end{pmatrix}, \tag{2.46}$$

which we can write as a linear combination of $\mathfrak{sl}(2,\mathbb{R})$ generators as follows

$$\partial_m V_E V_E^{-1} = (-\frac{1}{2}\Delta^{-1}\partial_m\Delta)h + (\frac{1}{2}\Delta^{-1}\partial_m\tilde{\psi})(e+f) + (\frac{1}{2}\Delta^{-1}\partial_m\tilde{\psi})(e-f). \tag{2.47}$$

We recognize $(e-f) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \equiv X^1$ as the generator of $\mathrm{SO}(2)$, the maximal compact subgroup of $\mathrm{SL}(2,\mathbb{R})$, while the non-compact part is generated by $(e+f) \equiv X^2$ and $h \equiv X^3$. Comparing to relation (2.28), we find that

$$P_m^2 = \frac{1}{2}\Delta^{-1}\partial_m\tilde{\psi}, \quad P_m^3 = -\frac{1}{2}\Delta^{-1}\partial_m\Delta, \quad Q_m^1 = \frac{1}{2}\Delta^{-1}\partial_m\tilde{\psi} \tag{2.48}$$

with

$$P_m^2 X^2 + P_m^3 X^3 = P_m^a X^a \equiv P_m \quad \text{and} \quad Q_m^1 X^1 \equiv Q_m. \tag{2.49}$$

We can now write the Lagrangian (2.8) as

$$\mathcal{L}^{(3d)} = \sqrt{g_3}\left(R^{3d} - \frac{1}{2}g^{mn}\langle P_m, P_n\rangle\right) \tag{2.50}$$

where due to (2.38), which in this case translates to $\langle h, h\rangle = 4$, we have that in terms of the matrices $P_m$ given by (2.48),(2.49)

$$\langle P_m, P_n\rangle = 2\mathrm{Tr}(P_m P_n) = \Delta^{-2}(\partial_m\Delta\partial_n\Delta + \partial_m\tilde{\psi}\partial_n\tilde{\psi}). \tag{2.51}$$

Further reduction to two dimensions yields (2.12) which now takes the form

$$\mathcal{L}_E^{(2d)} = \sqrt{g_2}\rho\left(R^{(2d)} + 2g^{\mu\nu}f^{-1}\partial_\mu f\rho^{-1}\partial_\nu\rho - \frac{1}{2}g^{\mu\nu}\langle P_\mu, P_\nu\rangle\right). \tag{2.52}$$

The equations of motion derived from the above Lagrangian are

$$R_{\mu\nu}^{(2d)} - \frac{1}{2}g_{\mu\nu}R^{(2d)} = \frac{1}{2}\langle P_\mu, P_\nu\rangle - \frac{1}{4}g_{\mu\nu}\langle P, P\rangle - 2f_E^{-1}\partial_\mu f_E\rho^{-1}\partial_\nu\rho + + g_{\mu\nu}(f_E^{-1}\partial_\sigma f_E\rho^{-1}\partial^\sigma\rho) \tag{2.53a}$$

$$\partial_\mu(\sqrt{g_2}f_E^{-1}g^{\mu\nu}\partial_\nu f_E) = \frac{1}{2}\sqrt{g_2}\left(R^{(2d)} - \frac{1}{2}g^{\mu\nu}\langle P_\mu, P_\nu\rangle\right) \tag{2.53b}$$

$$\partial_\mu\left(\sqrt{g_2}g^{\mu\nu}\partial_\nu\rho\right) = 0 \tag{2.53c}$$

$$D_\mu(\rho P^\mu) = 0. \tag{2.53d}$$

These equations simplify further, since in two dimensions we can bring the base metric (locally) in a conformally flat form, i.e. $g_{\mu\nu} = \lambda^2(x)\delta_{\mu\nu}$ (in suitable coordinates). The factor $\lambda^2$ can be absorbed into the conformal factor $f_E$ and thus have $g_{\mu\nu} = \delta_{\mu\nu}$. Equation (2.53a), whose left hand side now vanishes, becomes a system of first order equations for $f_E$ while (2.53d) and (2.53c) become the same equations in flat space. In particular, $\rho$, satisfying equation $\Box\rho = 0$ is a harmonic function on $\mathbb{R}^2$. Together with the conjugate harmonic function $z$ defined by

$$\partial_\mu\rho + {}^*\partial_\mu z = 0, \tag{2.54}$$

where ${}^*\partial_\mu z = \epsilon_{\mu\nu}\partial^\nu z$, the pair $(\rho, z)$ are Weyl canonical coordinates for the two-dimensional base manifold, provided that $\partial_\mu\rho \neq 0$ almost everywhere. The revised set of equations can be written in a convenient form by combining $(\rho, z)$ to form the complex variables $x^\pm = \frac{1}{2}(z \mp i\rho)$

$$\pm i f_E^{-1}\partial_\pm f_E = \frac{\rho}{4}\langle P_\pm P_\pm\rangle \tag{2.55a}$$

$$D_\mu(\rho P^\mu) = 0 \tag{2.55b}$$

Finally, equation (2.53b) is omitted since it is fulfilled by virtue of (2.53a) with the choice of Weyl coordinates.

We note that for the symmetric space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, the involutive automorphism that leaves the $\mathrm{SO}(2)$ subgroup invariant acts on matrices in $\mathrm{SL}(2,\mathbb{R})$ as

$$\tau\,:\,g \to \eta^{-1}\left(g^T\right)^{-1}\eta \quad \text{with} \quad \eta = \mathrm{diag}(1,1) \tag{2.56}$$

where $\eta$ is the invariant metric of SO(2). The induced Lie algebra automorphism acts as

$$\tau\,:\,X\to-\eta^{-1}X^T\eta,\quad X\in\mathfrak{sl}(2,\mathbb{R}). \tag{2.57}$$

In terms of the anti-involution "$\sharp$", we have that

$$g^\sharp=\tau(g^{-1})=\eta^{-1}g^T\eta,\quad g\in\mathrm{SL}(2,\mathbb{R}), \tag{2.58}$$

$$X^\sharp=-\tau(X)=\eta^{-1}X^T\eta,\quad X\in\mathfrak{sl}(2,\mathbb{R}). \tag{2.59}$$

### 2.2.2 The *Matzner-Misner* SL$(2,\mathbb{R})$/SO$(1,1)$ $\sigma$-model

As we have seen before, the two-dimensional gravity-matter system admits two equivalent descriptions, namely (2.12) and and (2.17). Similarly to the Ehlers Lagrangian, the Matzner-Misner one also admits a $\sigma$-model description. Indeed, (2.17) can be written as

$$\mathcal{L}^{(2d)}_{MM}=\sqrt{g_2}\rho\left(R^{(2d)}+2g^{\mu\nu}f^{-1}_{MM}\partial_\mu f_{MM}\rho^{-1}\partial_\nu\rho-\frac{1}{2}g^{\mu\nu}\langle\hat{P}_\mu,\hat{P}_\nu\rangle\right), \tag{2.60}$$

where $\hat{P}_\mu$ is now associated to the triangular element $V_{MM}$

$$V_{MM}(x)=\begin{pmatrix}\hat{\Delta}^{-1/2} & \psi\hat{\Delta}^{-1/2}\\ 0 & \hat{\Delta}^{1/2}\end{pmatrix} \tag{2.61}$$

where $\hat{\Delta}=\frac{\rho}{\Delta}$. In this case, the target space of the $\sigma$-model is the coset SL$(2,\mathbb{R})$/SO$(1,1)$ with invariant metric

$$ds^2_\sigma=\hat{\Delta}^{-2}(d\hat{\Delta}^{-2}-d\psi^2)=\langle\hat{P},\hat{P}\rangle. \tag{2.62}$$

The field equations from (2.60) read

$$\pm i f^{-1}_{MM}\partial_\pm f_{MM}=\frac{\rho}{4}\langle\hat{P}_\pm\hat{P}_\pm\rangle \tag{2.63a}$$

$$D_\mu(\rho\hat{P}^\mu)=0 \tag{2.63b}$$

$$\tag{2.63c}$$

in a very similar way to the ones from the Ehlers Lagrangian (2.52).

The involutive automorphism fixing the subgroup SO$(1,1)$ acts on matrices in SL$(2,\mathbb{R})$ as

$$\tau\,:\,g\to\bar{\eta}^{-1}\left(g^T\right)^{-1}\bar{\eta}\quad\text{with}\quad g\in\mathrm{SL}(2,\mathbb{R}),\,\bar{\eta}=\mathrm{diag}(-1,1), \tag{2.64}$$

$$\tau\,:\,X\to-\bar{\eta}^{-1}X^T\bar{\eta},\quad X\in\mathfrak{sl}(2,\mathbb{R}) \tag{2.65}$$

where $\bar{\eta}$ is the invariant metric of SO$(1,1)$. In terms of the anti-involution "$\sharp$", we have that

$$g^\sharp=\tau(g^{-1})=\bar{\eta}^{-1}g^T\bar{\eta},\quad g\in\mathrm{SL}(2,\mathbb{R}), \tag{2.66}$$

$$X^\sharp=-\tau(X)=\bar{\eta}^{-1}X^T\bar{\eta},\quad X\in\mathfrak{sl}(2,\mathbb{R}). \tag{2.67}$$

## 2.3 The Geroch group

In this section, we will discuss the Geroch symmetry [3], arising from the combination of the Ehlers and Matzner-Misner symmetries. By virtue of the duality relation (2.20), these symmetries combine and give rise to the enlarged Geroch symmetry, acting on the space of solutions of two-dimensional gravity. Its understanding in group theoretical terms came later than its discovery and provided new insights as well as a new approach to solution generation through the implementation of the Geroch group. We will devote the subsequent chapters to the latter, practical aspect of the Geroch group, but before that, we will provide some introduction on the underlying mathematical structure of this symmetry.

Let us start with the Ehlers coset element $V_E$

$$V_E = \begin{pmatrix} \Delta^{-1/2} & \tilde{\psi}\Delta^{-1/2} \\ 0 & \Delta^{1/2} \end{pmatrix} \tag{2.68}$$

as defined in (2.43). We will now find out how the fields $(\Delta, \tilde{\psi})$ change as a result of an $\mathrm{SL}(2,\mathbb{R})_E$ transformation. Consider the transformation rule (2.24) and take its infinitesimal form

$$\delta V_E = V_E \delta g + \delta k V_E. \tag{2.69}$$

Denoting the $\mathfrak{sl}(2,\mathbb{R})_E$ -generators as $(e_1, h_1, f_1)$, we take $\delta g$ with $g \in \mathrm{SL}(2,\mathbb{R})_E$ to be of the form $(\alpha_1 e_1 + \alpha_2 h_1 + \alpha_3 f_1)$ with $(\alpha_1, \alpha_2, \alpha_3)$ constant parameters of the transformation and the local transformation $\delta k$ with $k \in \mathrm{SO}(2)$ to be of the form $(\omega(e_1 - f_1))$ where $\omega$ depends on the fields. We get from (2.69):

$$\begin{aligned} \delta V_E &= V_E(\alpha_1 e_1 + \alpha_2 h_1 + \alpha_3 f_1) + (\omega(e_1 - f_1))V_E = \\ &= \begin{pmatrix} \alpha_2\Delta^{-1/2} + \alpha_3\tilde{\psi}\Delta^{-1/2} & \alpha_1\Delta^{-1/2} - \alpha_2\tilde{\psi}\Delta^{-1/2} + \omega\Delta^{1/2} \\ \alpha_3\Delta^{1/2} - \omega\Delta^{-1/2} & -\alpha_2\Delta^{1/2} - \omega\tilde{\psi}\Delta^{-1/2} \end{pmatrix}. \end{aligned} \tag{2.70}$$

We see that in order for the local transformation $k$ to restore the triangular form of $V_E$, we must have that

$$\alpha_3\Delta^{1/2} - \omega\Delta^{-1/2} = 0 \Rightarrow \omega = \alpha_3\Delta. \tag{2.71}$$

Substituting $\omega$ in (2.70) and comparing to the variation of $V_E$ with respect to the fields $(\Delta, \tilde{\psi})$

$$\delta V_E = \begin{pmatrix} -\frac{1}{2}\Delta^{-3/2}\delta\Delta & -\frac{1}{2}\tilde{\psi}\Delta^{-3/2}\delta\Delta + \Delta^{-1/2}\delta\tilde{\psi} \\ 0 & \frac{1}{2}\Delta^{-1/2}\delta\Delta \end{pmatrix}, \tag{2.72}$$

we find the variations $\delta\Delta, \delta\tilde{\psi}$ due to the transformation (2.69), namely

$$\delta\Delta = -2\alpha_2\Delta - 2\alpha_3\tilde{\psi}\Delta \tag{2.73}$$

$$\delta\tilde{\psi} = \alpha_1 - 2\alpha_2\tilde{\psi} - \alpha_3(\tilde{\psi}^2 - \Delta^2). \tag{2.74}$$

From the above variations we infer that the generator $e_1$ leaves $\Delta$ invariant and shifts $\tilde{\psi}$ by a constant. Both $\Delta, \tilde{\psi}$ are rescaled by the action of the generator $h_1$, while $f_1$ induces a non-linear transformation in the fields, as we can see from the terms proportional to $\alpha_3$ in (2.73),(2.74). This non-linear transformation is often called the "Ehlers transformation" .

Very similar calculations can be done for the action of the Matzner-Misner $\mathrm{SL}(2,\mathbb{R})$ group on the fields $\hat{\Delta} = \frac{\rho}{\Delta}$ (or $\Delta$), $\psi$. We note that the $\rho$ is not acted upon by the group transformation, that is $\delta\rho = 0$. We get

$$\delta\hat{\Delta} = -2\alpha_2'\hat{\Delta} - 2\alpha_3'\psi\hat{\Delta} \qquad \text{or} \quad \big(\delta\Delta = 2\alpha_2'\Delta + 2\alpha_3'\psi\Delta\big) \tag{2.75}$$

$$\delta\psi = \alpha_1' - 2\alpha_2'\psi - \alpha_3'(\psi^2 + \frac{\rho^2}{\Delta^2}), \tag{2.76}$$

where $\alpha_1', \alpha_2', \alpha_3'$ are the constant parameters of the transformation, associated to the $\mathfrak{sl}(2,\mathbb{R})_{MM}$ -generators $(e_0, h_0, f_0)$. We see again that the generator $f$ acts non-linearly while the rest of the transformations induce shifts and rescalings of the fields $\Delta, \psi$. We note that, in this case, the local transformation $\delta k$ is taken with $k \in \mathrm{SO}(1,1)$.

Recall that, although we have shown that two-dimensional gravity admits two distinct $\sigma$-model descriptions, with distinct symmetries, there is a relation between the respective component fields:

$$\begin{aligned} \Delta &\longleftrightarrow \frac{\rho}{\Delta} \\ \tilde{\psi} &\longleftrightarrow \psi \\ f_E &\longleftrightarrow f_{MM}, \end{aligned}$$

which is referred to as the Kramer-Neugebauer transformation. Moreover, the fields $\tilde{\psi}, \psi$ are related to each other through the duality relation (2.20). This prompts the question of how the symmetry of each system acts on the other. For example, one can ask how the Ehlers group acts on $V_{MM}$ and $f_{MM}$. Starting from (2.20), we vary and find (again, we take the variation $\delta\rho$ to be zero)

$$\begin{aligned} {}^*\partial_\mu(\delta\tilde{\psi}) &= \delta(\rho^{-1}\Delta^2\partial_\mu\psi) \Rightarrow \\ \Rightarrow \partial_\mu(\delta\psi) &= \rho^*\left(\Delta^{-2}\partial_\mu(\delta\tilde{\psi}) - 2\Delta^{-3}\partial_\mu\tilde{\psi}\delta\Delta\right). \end{aligned} \tag{2.77}$$

Using (2.73),(2.74 in the above equation, we arrive at the variation of $\psi$ due to the transformation (2.69). We find

$$\partial_\mu(\delta\psi) = \rho^*\left(\Delta^{-2}\partial_\mu(\alpha_1) + 2\alpha_2\Delta^{-2}\partial_\mu\tilde{\psi} + 2\alpha_3(\Delta^{-2}\tilde{\psi}\partial_\mu\tilde{\psi} + \Delta^{-1}\partial_\mu\Delta)\right). \tag{2.78}$$

From the above equation, we see that the generator $e_1$ has the effect of shifting $\psi$ by a constant

$$e_1 : \quad \partial_\mu(\delta\psi) = 0 \Rightarrow \delta\psi = \text{const.} \equiv c_1 \tag{2.79}$$

while $h_1$ induces the change

$$h_1 : \quad \partial_\mu(\delta\psi) = 2\partial_\mu\psi \Rightarrow \delta\psi = 2\psi + \text{const.}\,, \tag{2.80}$$

where we used (2.20) to substitute for ${}^*\partial_\mu\tilde{\psi}$ in the right hand side. Notice that $\psi$ is rescaled with an opposite sign to $\tilde{\psi}$ under the action of the generator $h$ (c.f. (2.74)). Finally, the variation of $\psi$ due to $f$, revealed by the terms proportional to $\alpha_3$ in (2.78), is more complicated. We have that

$$\begin{aligned} f_1 : \quad & \partial_\mu(\delta\psi) = 2\rho\,{}^*\left(\Delta^{-2}\tilde{\psi}\partial_\mu\tilde{\psi} + \Delta^{-1}\partial_\mu\Delta\right) \Rightarrow \\ & \Rightarrow {}^*\partial_\mu(\delta\psi) = -2\rho\left(\Delta^{-2}\tilde{\psi}\partial_\mu\tilde{\psi} + \Delta^{-1}\partial_\mu\Delta\right) \Rightarrow \\ & \Rightarrow {}^*\partial_\mu(\delta\psi) = {}^*\partial_\mu(-2\varphi_1), \end{aligned} \tag{2.81}$$

where the function $\varphi_1$ is defined as

$${}^*\partial_\mu\varphi_1 = \rho\left(\Delta^{-2}\tilde{\psi}\partial_\mu\tilde{\psi} + \Delta^{-1}\partial_\mu\Delta\right), \tag{2.82}$$

which results in the change of $\psi$ under the action of the $f$ $\mathrm{SL}(2,\mathbb{R})_E$ generator

$$\delta\psi = -2\varphi_1\,. \tag{2.83}$$

It is important to note that the action of $f$, unlike that of $e, h$, is non-linear in the fields and non-local: one needs to integrate (2.82) to determine the function $\varphi_1$. Moreover, one can proceed to evaluate the action of $\mathrm{SL}(2,\mathbb{R})_E$ on $\varphi_1$; in that process, yet a another function $\varphi_2$ is generated (again this feature appears in the change of $\varphi_1$ under the action of generator $f$). This process does not stop after finitely many steps. The integrability conditions for the new functions -or potentials as they are often called- follow from the equation of motion (2.55b).

For the conformal factor $f_{MM}$, we find that

$$f_{MM}^{-1}\delta f_{MM} = -\frac{1}{2}\Delta^{-1}\delta\Delta \tag{2.84}$$

which follows from $f_{MM} = f_E\rho^{1/4}\Delta^{-1/2}$, where $f_E$ is invariant under these transformations. Therefore, under an infinitesimal $\mathrm{SL}(2,\mathbb{R})_E$ transformation, $f_{MM}$ varies as

$$\text{due to}\, h_1 : \, \delta f_{MM} = f_{MM}, \quad \text{due to}\, f_1 : \, \delta f_{MM} = \tilde{\psi} f_{MM}, \tag{2.85}$$

while $e_1$ induces no change on $f_{MM}$.

Similarly, we examine the action of the Matzner-Misner transformations on the Ehlers data. Starting again from relation (2.20), we get

$$\partial_\mu(\delta\tilde{\psi}) = -\rho^{-1}\,{}^*\left(2\Delta\partial_\mu\psi\delta\Delta + \Delta^2\partial_\mu(\delta\psi)\right), \tag{2.86}$$

where we substitute the variation (2.75),(2.76) of the fields $\Delta, \psi$ on the right hand side and obtain

$$\partial_\mu(\delta\tilde{\psi}) = -\rho^{-1\,*}\left(\Delta^2\partial_\mu(\alpha_1') + 2\alpha_2'\Delta^2\partial_\mu\psi - 2\alpha_3'\Delta^2\psi\partial_\mu\psi + 2\alpha_3'\Delta\rho\partial_\mu\left(\frac{\rho}{\Delta}\right)\right). \tag{2.87}$$

From the above expression, we infer that

$$\text{due to}\, e_0 : \; \delta\tilde{\psi} = \text{const.} \equiv c_0\,, \qquad \text{due to}\, h_0 : \; \delta\tilde{\psi} = 2\tilde{\psi} + \text{const.} \tag{2.88}$$

and the variation of $\tilde{\psi}$ due to $f_0$ is once again more involved. We find that

$$\partial_\mu(\delta\tilde{\psi}) = 2\,{}^*\partial_\mu\hat{\varphi}_1, \tag{2.89}$$

with ${}^*\partial_\mu\hat{\varphi}_1 = \rho^{-1\,*}\left(\Delta^2\psi\partial_\mu\psi - \Delta\rho\partial_\mu\left(\frac{\rho}{\Delta}\right)\right)$. As was the case before, the introduction of a new function $\hat{\varphi}_1$ is required and yet a new one will be generated by the action of $\mathrm{SL}(2,\mathbb{R})_{MM}$ on it ; this process generates infinitely many new functions.

**Infinite-dimensional symmetry**

Understanding the interplay of the $\mathrm{SL}(2,\mathbb{R})_E$ and $\mathrm{SL}(2,\mathbb{R})_{MM}$ symmetries leads to the concept of infinite-dimensional symmetries. From the group theory point of view, this infinite-dimensional group of transformations is associated to a Lie algebra of Kac-Moody type [25, 38]. Such algebras are built by $\mathfrak{sl}(2)$-triples $(h_i, e_i, f_i)$ satisfying the commutation relations

$$\begin{aligned} [h_i, h_j] &= 0 \\ [h_i, e_j] &= A_{ij}e_j \\ [h_i, f_j] &= -A_{ij}f_j \\ [e_i, f_j] &= \delta_{ij} \end{aligned} \tag{2.90}$$

and the Serre relations

$$[e_i, [e_i, ...[e_i, e_j]...]] = (\mathrm{ad}\, e_i)^{1-A_{ij}}(e_j) = 0 \tag{2.91}$$

$$[f_i, [f_i, ...[f_i, f_j]...]] = (\mathrm{ad}\, f_i)^{1-A_{ij}}(f_j) = 0, \tag{2.92}$$

where $A_{ij}$ are the entries of the Cartan matrix, with properties

$$A_{ij} \in \mathbb{Z}\,, \quad A_{ii} = 2\,, \quad A_{ij} \leq 0 \;\; \forall\, i \neq j. \tag{2.93}$$

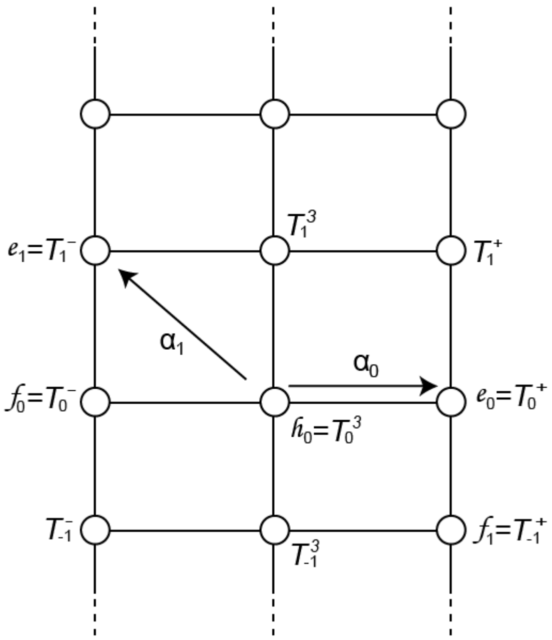
In the case of the Geroch symmetry of two-dimensional gravity, the associated Kac-Moody algebra is generated by the two copies of $\mathfrak{sl}(2,\mathbb{R})$ generators $e_0, h_0, f_0$ and $e_1, h_1, f_1$. The Cartan matrix reads

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{2.94}$$

The relation of this construction to the group of transformations arising from the combined action of $\mathrm{SL}(2,\mathbb{R})_E$ and $\mathrm{SL}(2,\mathbb{R})_{MM}$ is established if we identify

$$\begin{aligned} e_0 &= T_0^+, \quad f_0 = T_0^-, \quad h_0 = T_0^3 \\ e_1 &= T_1^-, \quad f_1 = T_{-1}^+, \quad h_1 = c - T_0^3, \end{aligned} \tag{2.95}$$

with $c = h_0 + h_1$ the central element that commutes with all Lie algebra elements, given by $c = \sum_{k=0}^{1} = n_k h_k$, with $n$ the zero eigenvector of $A$ (here $n = (1,1)$). Note that the lower indices in the $T^\pm, T^3$ generators in (2.95) now refer to levels of the affine algebra.




The full affine Kac-Moody algebra includes an additional generator $d$ that extends the Cartan subalgebra to $\{h_0, c\} \oplus \{d\}$ and is defined by

$$[d, T_m^a] = m T_m^a, \tag{2.96}$$

with $m = ... -1, 0, 1, ...$ and $a = \pm, 3$. It is the inclusion of the element $d$ that gives rise to the two-dimensional root diagram above; the simple roots are $\alpha_0 = \sqrt{2}(1,0)$, $\alpha_1 = \sqrt{2}(-1,1)$ where the second component is the eigenvalue $m$ of $d$. The scalar product in the root space is defined as $\langle \alpha_i | \alpha_j \rangle = \alpha_i^\mu \alpha_j^\nu G_{\mu\nu} = A_{ij}$ with $G_{\mu\nu} = \mathrm{diag}\{1, 0\}$, such that the products $\langle \alpha_i | \alpha_j \rangle$, $i, j = 0, 1$ reproduce the entries of the Cartan matrix (2.94).

The commutation relations defining this Kac-moody algebra read

$$\begin{aligned} [h_0, e_{0,1}] &= \pm 2 e_{0,1}\,, \qquad [h_0, f_{0,1}] = \mp 2 f_{0,1}, \\ [h_1, e_{0,1}] &= \mp 2 e_{0,1}\,, \qquad [h_1, f_{0,1}] = \pm 2 f_{0,1} \end{aligned} \tag{2.97}$$

as well as

$$[h_i, h_j] = 0\,, \qquad [e_i, f_j] = \delta_{ij} h_j \tag{2.98}$$

$$[e_i, [e_i, [e_i, e_j]]] = 0\,, \qquad [f_i, [f_i, [f_i, f_j]]] = 0, \tag{2.99}$$

with $i, j = 0, 1$.

We can see how the above symmetry is realized through the field transformations with a few examples, e.g.

$$\begin{aligned}
[h_0, e_1](\Delta) &= h_0 e_1(\Delta) - e_1 h_0(\Delta) = -2e_1(\Delta) \\
[h_0, e_1](\tilde{\psi}) &= h_0 e_1(\tilde{\psi}) - e_1 h_0(\tilde{\psi}) = -2e_1(\tilde{\psi}) \\
[h_0, f_1](\Delta) &= h_0 f_1(\Delta) - f_1 h_0(\Delta) = h_0(-2\tilde{\psi}\Delta) - f_1(2\Delta) = 2f_1(\Delta) \\
[h_0, f_1](\tilde{\psi}) &= h_0(\Delta^2 - \tilde{\psi}^2) - f_1(2\tilde{\psi}) = 4(\Delta^2 - \tilde{\psi}^2) - 2(\Delta^2 - \tilde{\psi}^2) = 2f_1(\tilde{\psi}) \\
[h_1, e_0](\psi) &= h_1 e_0(\psi) - e_0 h_1(\psi) = -e_0(2\psi) = -2e_0(\psi) \\
[h_1, f_0](\psi) &= h_1\left(-\psi^2 - \frac{\rho^2}{\Delta^2}\right) - f_0(2\psi) = -4\left(\psi^2 + \frac{\rho^2}{\Delta^2}\right) + 2\left(\psi^2 + \frac{\rho^2}{\Delta^2}\right) \\
&= 2f_0(\psi),
\end{aligned}$$

where we used the variations (2.73),(2.74),(2.75),(2.76) and the notation $T^a_m(\psi)$ means the variation of $\psi$ due to generator $T^a_m$ and similarly for the other scalars. We see that the above commutation relations agree with the structure (2.90) with Cartan matrix $(A_{ij})$ given by (2.94).

## 2.4 Integrability and the linear system

After reviewing the symmetry properties as well as the $\sigma$-model description of gravity reduced to two dimensions, we will focus the discussion on the equations of motion (2.55b) and their representation as a "Lax pair" or linear system. In this section, we will work with the Ehlers formulation only and so we will mostly drop the subscript "$E$", to make the notation simpler (we will come back to the explicit notation with the subscripts when it is needed). Working in the Ehlers coset has computational advantages when one applies solution generating transformations and facilitates the generalisation to larger symmetries of Ehlers type found in supergravity.

We have seen earlier that the equations of reduced gravity include a system of equations for the conformal factor as well as the non-linear $\sigma$-model equations (2.55b). We will focus our attention on the latter, and will turn to (2.55a) once $P_\mu$ is known; the conformal factor $f_E$ can then be obtained by single integration of (2.55a).

The strategy in dealing with (2.55b) is to find a linear system of equations which implicitly represent it, in the sense that the compatibility conditions for this linear system coincide with the $\sigma$-model equation of motion. To present this construction, let us start our way to the linear system from the Geroch symmetry that we analysed above. We have seen previously that applying the Ehlers $\mathrm{SL}(2,\mathbb{R})$ symmetry on the Matzner-Misner scalars requires the introduction of new scalars. These in turn give rise to new ones through the same group action and so on ad infinitum. At the level of the equations of motion, these new scalars give rise to new currents which must be of a form that preserves covariance under the Matzner-Misner group. Therefore, we will take them to be linear combinations of the current $P_\mu = D_\mu V V^{-1}$ and its dual ${}^*P_\mu$ :

$$\tilde{J}_\mu = a D_\mu V V^{-1} + b\,{}^*(D_\mu V V^{-1}), \tag{2.100}$$

where $\tilde{J}_\mu$ represents the "generalised current" constructed by the original as well as the new scalars. In order to resolve the structure of the left hand side in (2.100), let us introduce a function $\mathcal{V}(t,x)$ of the form

$$\mathcal{V}(t,x) = V_0 + tV_1 + \frac{1}{2}t^2 V_2 + ... \tag{2.101}$$

such that it accommodates all the required scalars and also depends on a parameter $t$, that we will call the spectral parameter. The function $\mathcal{V}(t,x)$ is related to the familiar $V(x)$ through the limit

$$\lim_{t\to 0} \mathcal{V}(t,x) = V_0 \equiv V(x). \tag{2.102}$$

With the above ingredients, we make an ansatz that "generalises" (2.28)

$$\partial_\mu \mathcal{V}\mathcal{V}^{-1} = Q_\mu + aP_\mu + b\,{}^*P_\mu, \tag{2.103}$$

where $a, b$ are functions of $t$ to be specified shortly. Writing (2.103) as $\partial_\mu \mathcal{V} = (Q_\mu + aP_\mu + b\,{}^*P_\mu)\,\mathcal{V}$ and taking the integrability condition $\partial_\nu\partial_\mu\mathcal{V} = \partial_\mu\partial_\nu\mathcal{V}$, we arrive at

$$\begin{aligned}
&(a^2-1)[P_\mu, P_\nu] + b^2[{}^*P_\mu, {}^*P_\nu] + \frac{b}{\rho}\left(D_\nu\left(\rho\,{}^*P_\mu\right) - D_\mu\left(\rho\,{}^*P_\nu\right)\right) + \\
&+ \rho\left(\partial_\nu\left(\frac{b}{\rho}\right){}^*P_\mu - \partial_\mu\left(\frac{b}{\rho}\right){}^*P_\nu\right) + ({}^*\partial_\nu a)\,{}^*P_\mu - ({}^*\partial_\mu a)\,{}^*P_\nu = 0.
\end{aligned} \tag{2.104}$$

To reach the above relation, we have used the equations

$$\partial_\nu Q_\mu - \partial_\mu Q_\nu + [Q_\mu, Q_\nu] = -[P_\mu, P_\nu] \tag{2.105}$$

$$D_\mu P_\nu - D_\nu P_\mu = 0\,, \tag{2.106}$$

arising from the integrability condition $\partial_\nu\partial_\mu V = \partial_\mu\partial_\nu V$ associated to (2.28). We have also used that $(\partial_\mu a)\,P_\nu = -\,({}^*\partial_\nu a)\,{}^*P_\mu$. Moreover, using the $\sigma$-model equation of motion (2.55b) together with $[{}^*P_\mu, {}^*P_\nu] = [P_\mu, P_\nu]$ in (2.104), we find that it holds when the following equations are satisfied

$$a^2 + b^2 - 1 = 0 \tag{2.107}$$

and

$$\rho\partial_\mu\left(\frac{b}{\rho}\right) + {}^*\partial_\mu a = 0. \tag{2.108}$$

Setting $t = -\frac{b}{1+a}$, equation (2.107) gives

$$a = \frac{1-t^2}{1+t^2}, \qquad b = \frac{-2t}{1+t^2}\,. \tag{2.109}$$

With these expressions for $a, b$, equation (2.108) becomes a differential equation for $t$ :

$$\partial_\mu t = \frac{t}{\rho(1+t^2)}\left((1-t^2)\partial_\mu\rho - 2t\,{}^*\partial_\mu\rho\right) \tag{2.110}$$

which can be brought to the form

$$\partial_\mu\left(\rho\left(\frac{1}{t} - t\right)\right) = 2\,{}^*\partial_\mu\rho = -2\partial_\mu z, \tag{2.111}$$

using relation (2.54). Integrating (2.111), we obtain

$$\frac{1}{t} - t = \frac{2}{\rho}(w - z), \tag{2.112}$$

with $w$ an integration constant that will play the role of an $x$-independent spectral parameter. For each point $(z, \rho)$, $\rho \neq 0$ on the two-dimensional base manifold, equation (2.112) represents a non-singular, two-sheeted Riemann surface with branch points at $w = z + i\rho$. For each $w$, the two roots of (2.112) are

$$t_\pm = \frac{1}{\rho}\left((z-w) \pm \sqrt{(z-w)^2 + \rho^2}\right) = -\frac{1}{t_\mp} \tag{2.113}$$

with the transformation $t \to -\frac{1}{t}$ exchanging the two sheets. For our purposes, we will take $w$ to be real and choose the solution $t_+$ as the physical sheet on which the spectral parameter lives.

With the above considerations, we have reached the final form of the system of linear equations (2.103), that is [13, 31, 32]

$$\partial_\mu\mathcal{V}\mathcal{V}^{-1} = Q_\mu + \frac{1-t^2}{1+t^2}P_\mu - \frac{2t}{1+t^2}\,{}^*P_\mu. \tag{2.114}$$

We will call this the BM linear system, after Breitenlohner and Maison who first analysed it in [13]. Using the complex coordinates $x^\pm = \frac{1}{2}(z \mp i\rho)$ we can bring this system in the convenient form [13, 21, 32]

$$\partial_\pm\mathcal{V}\mathcal{V}^{-1} = Q_\pm + \frac{1 \mp it}{1 \pm it}P_\pm\,, \tag{2.115}$$

that we will be using from now on.

Thus far we have shown that the non-linear $\sigma$-model equation is integrable, by means of the linear system (2.115). In the process of determining this system, we introduced a new parameter $t$ as well as a $t$-dependent function $\mathcal{V}(t,x)$ containing all additional scalars generated by the combined symmetries of the original equation of motion. At this point, one could ask what is the group theoretic interpretation of the new objects. The spectral parameter $t$, that is known to appear in Lax pairs of integrable systems in general, here has an additional meaning. In fact, the affine extension of the Ehlers group $G_E$ to the infinite group of Geroch transformations requires an additional parameter on which the Geroch elements depend. Indeed, the affine Lie algebra associated to the finite algebra $\mathfrak{g}_E$ can be realised as an algebra of polynomial maps from $\mathbb{C}$ into $\mathfrak{g}_E$ together with a central charge $c$. Taking $T^a$ to be the generators of $\mathfrak{g}_E$, the Lie bracket on the affine algebra is defined as

$$[T^a_m, T^b_n] = f^{ab}_c T^c_{m+n} + mc\delta_{m+n,0}\delta^{ab} \tag{2.116}$$

with $m \in \mathbb{Z}$. The set of maps from $\mathbb{C}$ to $\mathfrak{g}_E$ include the maps from the unit circle into $\mathfrak{g}_E$ and these form the loop algebra $\widehat{\mathfrak{g}_E}$. [4] With these considerations, at the group level, we may view the function $\mathcal{V}(t,x)$ that generalises the finite coset representative $V(x)$ as the analogous object in the affine group. Indeed, the element $\mathcal{V}(t,x)$ is chosen in a generalised "triangular" gauge, in the sense that it admits a series expansion in $t$ and the $t$-independent term is the triangular element $V(x)$ (see (2.101),(2.102)). We can write the transformation property for $\mathcal{V}(t,x)$ analogous to (2.24), that is

$$\mathcal{V}(t) \to k(t)\mathcal{V}(t)g(w), \tag{2.117}$$

(suppressing the $x$-dependence). The function $g(w) : \mathbb{C} \to G_E$ is a global transformation in the affine group ($w$ is the $x$-independent spectral parameter that we encountered in (2.112)). We impose that $g(w)$ is a function holomorphic around $w = \infty$, so that it admits an expansion as in (2.101). The transformation $k(t)$ depends on the $x$-dependent spectral parameter $t$ and plays again the role of gauge compensator; it is chosen such that $\mathcal{V}(t)$ remains "triangular" in the sense of (2.101).

Next, we may extend the generalisation to the symmetric space automorphism $\tau$, that we denote $\tau^\infty$. Using again the $\sharp$ symbol for the generalised anti-involution, the action on $t$-dependent functions such as $\mathcal{V}(t,x)$ is defined as

$$(\mathcal{V}(t,x))^\sharp = \mathcal{V}^\sharp\left(-\frac{1}{t}, x\right). \tag{2.118}$$

At the Lie algebra level, applying $\tau^\infty$ to equation (2.115), reveals that $\partial_\mu \mathcal{V}\mathcal{V}^{-1}$

[4] We may recognize the affine algebra generated by $\mathfrak{sl}(2,\mathbb{R})_E, \mathfrak{sl}(2,\mathbb{R})_{MM}$ discussed in the previous section as $\widehat{\mathfrak{sl}(2,\mathbb{R})} \oplus \mathbb{C}$.

remains invariant, or equivalently $\sharp$ -anti-invariant. Indeed we have that

$$\begin{aligned}\left(\partial_\pm \mathcal{V}\mathcal{V}^{-1}\right)^\sharp &= \left(Q_\pm + \frac{1\mp it}{1\pm it}P_\pm\right)^\sharp = Q_\pm^\sharp + \frac{1\mp i(-1/t)}{1\pm i(-1/t)}P_\pm^\sharp \\ &= -Q_\pm - \frac{1\mp it}{1\pm it}P_\pm = -\left(Q_\pm + \frac{1\mp it}{1\pm it}P_\pm\right), \end{aligned} \tag{2.119}$$

where the $\sharp$ acts on $Q_\pm\,, P_\pm$ as in (2.29). The anti-invariance of the Lie algebra element $\partial_\pm \mathcal{V}\mathcal{V}^{-1}$ means that it belongs to the "compact" part of the affine algebra, if we consider a splitting analogous to the finite algebra. From property (2.119), it is easy to deduce that if $\mathcal{V}(t)$ is a solution to the linear system (2.115), then $(\mathcal{V}(t,x))^{\sharp\,-1}$ is also a solution to the same differential equations; the respective ($w$-dependent) integration constants will be different however [5].

In analogy to the matrix $M$ defined in (2.33), we introduce the monodromy matrix $\mathcal{M}$ given by

$$\mathcal{M} = \mathcal{M}(w) = (\mathcal{V}(t))^\sharp\, \mathcal{V}(t) = \mathcal{V}^\sharp\left(-\frac{1}{t}\right)\mathcal{V}(t). \tag{2.120}$$

Now taking the derivative of (2.120), we find that $\partial_\mu \mathcal{M} = 0$,

$$\begin{aligned}\partial_\mu \mathcal{M} &= \partial_\mu\left(\mathcal{V}^\sharp \mathcal{V}\right) = \\ &= \mathcal{V}^\sharp \mathcal{V}^{-1\,\sharp}\left(\partial_\mu \mathcal{V}\right)^\sharp \mathcal{V} + \mathcal{V}^\sharp \partial_\mu \mathcal{V}\mathcal{V}^{-1}\mathcal{V} = \\ &= \mathcal{V}^\sharp\left(\left(\partial_\mu \mathcal{V}\mathcal{V}^{-1}\right)^\sharp + \partial_\mu \mathcal{V}\mathcal{V}^{-1}\right)\mathcal{V} = 0, \end{aligned} \tag{2.121}$$

since $\left(\partial_\mu \mathcal{V}\mathcal{V}^{-1}\right)^\sharp = -\partial_\mu \mathcal{V}\mathcal{V}^{-1}$, as shown above. This justifies the definition of $\mathcal{M} = \mathcal{M}(w)$ as a function of $w$ only. Moreover, the function $\mathcal{M}(w)$ is by construction $\sharp$-invariant and it transforms under the enlarged group in a manner very similar to $\mathcal{M}$ in (2.34), i.e.

$$\mathcal{M}(w) \to \mathcal{M}^g(w) = g^\sharp(w)\mathcal{M}(w)g(w). \tag{2.122}$$

This property makes $\mathcal{M}(w)$ a very useful object to work with, especially in the context of solution generation techniques. Unlike $\mathcal{V}(t,x)$, the transformation of $\mathcal{M}$ does not involve a local transformation $k(t)$ that is generally hard to find. The process of generating new solutions through Geroch transformations, starts from a seed solution $(V, \mathcal{V})$ that is simple to find, (e.g. flat space) from which $\mathcal{M}$ is also constructed. Next, $\mathcal{M}$ transforms into $\mathcal{M}^g$ under a global Geroch transformation, thus giving rise to a new solution. Reaching this new solution however, requires the rather difficult step of factorising $\mathcal{M}^g$ into $\mathcal{V}^{g\,\sharp}\mathcal{V}^g$. This is a so-called Riemann-Hilbert problem that is generally hard to solve. In what follows, we will focus on the

[5] From equation $\left(\partial_\pm \mathcal{V}\mathcal{V}^{-1}\right)^\sharp = -\left(Q_\pm + \frac{1\mp it}{1\pm it}P_\pm\right)$ and using that $\partial_\pm \mathcal{V}\mathcal{V}^{-1} = -\mathcal{V}\partial_\pm \mathcal{V}^{-1}$ as well as $\partial_\pm(\mathcal{V}(-1/t,x)) = (\partial_\pm \mathcal{V})(-1/t,x)$, it follows that $(\mathcal{V}(t,x))^{\sharp\,-1}$ satisfies the same equations as $\mathcal{V}(t,x)$. Moreover, since for any arbitrary function of $w$ only we have that $\partial_\pm C(w) = 0$, the solutions $(\mathcal{V}(t,x))^{\sharp\,-1}$, $\mathcal{V}(t,x)$ will be related as $\mathcal{V}(t,x) = (\mathcal{V}(t,x))^{\sharp\,-1}C(w)$.

sector of solitonic transformations, which means that we will only take $\mathcal{M}^g(w)$ to be meromorphic functions with single poles in $w$. In this case, not only is it possible for the Riemann-Hilbert problem to be solved with a purely algebraic process [13, 26], but the class of solutions obtained in this way include black holes, colliding plane wanes, cosmological solutions and even fuzzball solutions in higher dimensions. To summarize the process, we outline the steps as

$$V \to \mathcal{V}(t) \to \mathcal{M}(w) \to \mathcal{M}^g(w) \to \mathcal{V}^g(t) \to V^g, \tag{2.123}$$

where the superscript $g$ indicates a transformed object. The very last step of finding a solution, requires the limit of $\mathcal{V}^g$ at $t \to 0$, which gives the new $V^g$, from which we finally read off the new profiles of the scalar fields.

In order to obtain a new solution and write down the full line element, we need the new conformal factor $f_E^g$ as well. This function is obtained through integration of (2.55$a$) once $V^g$ is known, but as was remarkably discovered by Breitenlohner and Maison, there is a place for it in the group theoretical picture too. The conformal factor is acted upon by the central extension of the affine group and Breitenlohner and Maison were able to reach an explicit formula for the transformation of the conformal factor involving the group 2-cocycle [13]. Based on the observation by Julia in [15] that the conformal factor transforms under infinitesimal transformations as a Lie algebra cocycle, BM found the "group version" of this transformation by showing that the central extension in the affine group can be defined through a group 2-cocycle $\Omega$. It is defined through the relation

$$\Omega(b,c) - \Omega(ab,c) + \Omega(a,bc) - \Omega(a,b) = 0 \quad \text{with } \; a,b,c \in \text{loop group } \widehat{G_E} \tag{2.124}$$

and helps define the central extension of $\widehat{G_E}$ if one considers pairs $(a,\alpha) \in \widehat{G_E} \times \mathbb{C}$ with multiplication law given by [6]

$$(a, e^{\alpha}) \circ (b, e^{\beta}) = \Big(ab, e^{\alpha+\beta+\Omega(a,b)}\Big)\,. \tag{2.125}$$

Therefore, the extended group acts on the pairs $(\mathcal{V}, f_E)$ through an extension of the transformation law (2.117) as follows

$$(\mathcal{V}, f_E^{-1}) \to (k(\mathcal{V},g),1) \circ (\mathcal{V}, f_E^{-1}) \circ (g, e^{\gamma})^{-1} \quad \text{for } \; (g,e^{\gamma})^{-1} \text{ in affine group.} \tag{2.126}$$

In accordance with (2.120), BM define the pair

$$(\mathcal{M}, \mu) \equiv (\mathcal{V}, f_E^{-1})^{\sharp} \circ (\mathcal{V}, f_E^{-1}) = \Big(\mathcal{M}, f_E^{-2} e^{\Omega(\mathcal{V}^{\sharp},\mathcal{V})}\Big)\,, \tag{2.127}$$

where the transformation law (2.126) has been used. Moreover, as shown in [13], it is not only $\mathcal{M}$ that is $x$-independent, but the whole pair $(\mathcal{M},\mu)$. As a result, the second argument in the final pair in (2.127) must be constant; this gives the formula

$$f_E^2 = c e^{\Omega(\mathcal{V}^{\sharp},\mathcal{V})}, \tag{2.128}$$

[6] More details on Lie algebra and group 2-cocycles can be found in Appendix B of [13]. Another useful reference is [39].

where the constant $c$ is determined from the asymptotic behaviour of the solution. With this beautiful cocycle formula, BM achieved to accommodate all the data for a full new solution into the Geroch picture in a purely group theoretic fashion. However, in our inverse scattering constructions with the BM method, we will not need to use this cocycle formula. We will focus on solitonic solutions, where the conformal factor can be obtained algebraically from a simple formula.

Finally, in order to complete the discussion on the symmetries of the two-dimensional systems discussed here, one needs to include the Virasoro algebra that is also shown to represent a symmetry of such systems [40, 41, 42]. It is in fact beyond the Geroch symmetry and it arises from arbitrary reparameterizations of the constant spectral parameter. These transformations together with the infinitesimal transformations in the affine group form a semi-direct product in the standard way. As it is not of crucial relevance to our work, we will not touch upon the Virasoro symmetry.

# Chapter 3

# Inverse scattering in Einstein Gravity

In this chapter, we will discuss the inverse scattering method in two-dimensional gravity, as was applied by Belinski-Zakharov and Breitenlohner-Maison. The two approaches differ in several aspects, but both employ the concept of inverse scattering, that is the determination of solutions of differential equations from known "scattering data". The inverse scattering transform has been successfully applied in several non-linear problems, such as the Korteweg-de Vries equation, Sine-Gordon, non-linear Schrödinger equation and others (see e.g. [43] for a discussion on inverse scattering method in soliton theory). All of these are completely integrable systems with infinite dimensional symmetry. Moreover, the inverse scattering problem can be stated as a Riemann-Hilbert factorization problem that becomes algebraically manageable in the soliton sector. Although the terms "soliton" and "scattering" do not have a literal interpretation in the application of inverse scattering to gravity, they are carried over to this setting and refer to the mathematically analogous objects that are encountered in the non-linear problem. The discussion in this chapter includes a brief review of the BZ method and the BM factorization algorithm in the soliton sector as was analysed in the unpublished work [26]. Given this basis, we present the work completed in collaboration with A. Kleinschmidt and A. Virmani [44]. In this work, we discuss the interrelations of these two approaches and construct the Kerr-NUT solution using the BM technique. We find that for the case of Einstein gravity, the BZ method remains more practical for solution generation. However the BM approach, as a group theoretic method, can be generalised and applied to gravity problems (e.g. in supergravity) involving other group symmetries - we will proceed in this direction in the following chapters.

## 3.1 Belinski-Zakharov method

In the late 70's, Belinski and Zakharov (BZ) managed to adjust the inverse scattering technique to the non-linear problem in two-dimensional gravity and devise a method to generate new solutions from known "seed" solutions [9],[10],[11]. They were the first to show that the two-dimensional Einstein field equations are integrable by means of a Lax pair or linear system. We will briefly outline their approach and give the relation of the (BZ) linear system to the one derived by Breitenlohner-Maison, discussed in the previous chapter. Even though we restrict the discussion to $D = 4$ in this chapter, the equations of motion have the same form for $D$-dimensional spacetimes with $D-2$ commuting Killing vectors [45] and it is only the dimensionality of the Killing part of the metric that changes. Therefore, the BZ method can in principle be generalised to dimensions higher than four [11],[46],[47],[48]. However, for reasons that will be explained soon, this method is most gainful in $D = 4$ and $D = 5$ dimensions.

Let us start with four-dimensional spacetime with two commuting Killing isometries generated by one timelike and one spacelike Killing vector (we will continue working with stationary, axisymmetric spacetimes), where the line element can be written as

$$ds^2 = e^{2\nu}\left(d\rho^2 + dz^2\right) + g_{ab}d\bar{x}^a d\bar{x}^b, \tag{3.1}$$

with $\rho, z$ the Weyl coordinates on the base metric, such that $\det g = -\rho^2$ and the indices $a, b$ refer to the (Killing) coordinates $\bar{x}^a = t, \phi$ [1]. The functions in the metric (3.1) depend on $x \equiv (\rho, z)$ only and the equations of motion come in two sets, one for the metric $g = (g_{ab})$ and one for the conformal factor $e^{2\nu}$. Starting with the equations for $g$

$$\partial_\mu\left(\rho\partial^\mu g g^{-1}\right) = 0 \tag{3.2}$$

and defining the matrices

$$U = \rho\left(\partial_\rho g\right) g^{-1}, \qquad V = \rho\left(\partial_z g\right) g^{-1}, \tag{3.3}$$

we write equation (3.2) as

$$\partial_\rho U + \partial_z V = 0 \tag{3.4}$$

and the equations for the function $\nu(\rho, z)$ in (3.1) read

$$\partial_\rho \nu = -\frac{1}{2\rho} + \frac{1}{8\rho}\mathrm{Tr}\left(U^2 - V^2\right) \tag{3.5a}$$

$$\partial_z \nu = \frac{1}{4\rho}\mathrm{Tr}\left(UV\right). \tag{3.5b}$$

We note that equation (3.4) for $g$ does not involve $\nu$; a full solution to the problem is obtained by solving for $g$ and subsequently integrating (3.5a),(3.5b) to get $\nu$.

In the spirit of the inverse scattering method, Belinski and Zakharov showed that equation (3.4) is the compatibility condition of the linear system

$$D_1\Psi = \frac{\rho V - \lambda U}{\lambda^2 + \rho^2}\Psi, \qquad D_2\Psi = \frac{\rho U + \lambda V}{\lambda^2 + \rho^2}\Psi, \tag{3.6}$$

[1] For historical reasons, the spectral parameter in the BM linear system presented in section 2.4 is denoted by $t$, as is the time coordinate. We will denote the coordinates $(t, \phi)$ collectively by $\bar{x}$. In this discussion, except for its appearance in the line element, $t$ will refer to the BM spectral parameter.

where $\lambda$ is the $x$-independent spectral parameter and $D_1, D_2$ are the composite differential operators

$$D_1 = \partial_z - \frac{2\lambda^2}{\lambda^2+\rho^2}\partial_\lambda, \qquad D_2 = \partial_\rho + \frac{2\lambda\rho}{\lambda^2+\rho^2}\partial_\lambda \tag{3.7}$$

such that $D_1 D_2 \Psi = D_2 D_1 \Psi$. The generating function $\Psi(\lambda, \rho, z)$ is related to $g$ as

$$\Psi|_{\lambda=0} = g. \tag{3.8}$$

The above relation reminds us of the one between the BM generating function $\mathcal{V}(t,x)$ and the coset representative $V(x)$ (cf. (2.101),(2.102)) in section 2.4. Moreover, (3.4) can be viewed as the $\sigma$-model equations of motion written in terms of the matrix $g \in \mathrm{GL}(2,\mathbb{R})$ instead of the unimodular coset metric $M$ (cf. (2.41b)). The metric $g$ is related to the Matzner-Misner coset metric $M_{MM} = V^{\sharp}_{MM}V_{MM}$ as follows [13]

$$g = \rho\bar{\eta}M_{MM}. \tag{3.9}$$

where $\bar{\eta} = \mathrm{diag}(-1,1)$, cf. section 2.2.2,(2.64). Using (3.9) in (3.4), we find that

$$0 = \partial_\mu\left(\rho\partial^\mu g g^{-1}\right) = \Box\rho + \partial_\mu\left(\rho\partial^\mu M_{MM}M^{-1}_{MM}\right) = \partial_\mu\left(\rho\partial^\mu M_{MM}M^{-1}_{MM}\right), \tag{3.10}$$

where we used the $\rho$-equation of motion $\Box\rho = 0$. Thus we recover the $\sigma$-model equation of motion $\partial_\mu\left(\rho\partial^\mu M_{MM}M^{-1}_{MM}\right) = 0$, as expressed in terms of the matrix $M_{MM}$ [2]. With relation (3.10) in mind, it follows that one can formulate the BZ method such that the generating function $\Psi$ gives the unimodular matrix $M$ at $\lambda = 0$ instead of $g$. Moreover, since the $\sigma$-model equations of motion in the Ehlers and Matzner-Misner coset are identical in form, the BZ linear system can also be phrased in the Ehlers language [47]. We will proceed in this direction in order to facilitate comparison to the BM linear system in the Ehlers formulation of section (2.4).

**BZ Ehlers linear system**

Taking the matrices $U, V$ to be $U = \rho\partial_\rho M_E M_E^{-1}$ , $V = \rho\partial_z M_E M_E^{-1}$, let us rewrite the BZ (Ehlers) linear system as

$$\partial_z\Psi_E = \frac{\rho V - \lambda U}{\lambda^2+\rho^2}\Psi_E\,, \qquad \partial_\rho\Psi_E = \frac{\rho U + \lambda V}{\lambda^2+\rho^2}\Psi_E, \tag{3.11}$$

where $\lambda$ is the spectral parameter that we take now to be *x-dependent* and the function $\Psi_E$ is the generating function that satisfies

$$\Psi_E(\lambda = 0, \rho, z) = M_E(\rho, z). \tag{3.12}$$

[2]Indeed, using (2.36) (together with trace cyclicity) to express the trace in (2.60) in terms of $M$, we find that equation of motion for $M$ reads: $\partial_\mu\left(\rho\partial^\mu M M^{-1}\right) = 0$.

Taking the integrability condition $\partial_\rho \partial_z \Psi_E = \partial_z \partial_\rho \Psi_E$ for (3.11), we get

$$\partial_\rho \left( \frac{\rho V - \lambda U}{\lambda^2 + \rho^2} \right) - \partial_z \left( \frac{\rho U + \lambda V}{\lambda^2 + \rho^2} \right) + \frac{1}{(\lambda^2 + \rho^2)^2} \left[ (\rho V - \lambda U), (\rho U + \lambda V) \right] = 0, \quad (3.13)$$

which is fulfilled when equations $\partial_\mu \left( \rho \partial^\mu M_E M_E^{-1} \right) = 0$ hold (the equation we wish to solve) and $\lambda$ obeys the differential equations

$$\partial_z \lambda = \frac{-2\lambda^2}{\lambda^2 + \rho^2}, \qquad \partial_\rho \lambda = \frac{2\rho\lambda}{\lambda^2 + \rho^2}. \quad (3.14)$$

The above equations justify the choice of BZ (3.7) for the differential operators $D_1, D_2$

$$D_1 = \partial_z - \frac{2\lambda^2}{\lambda^2 + \rho^2} \partial_\lambda = \partial_z|_{\lambda\,\text{fixed}} + \partial_z \lambda \partial_\lambda, \quad (3.15)$$

$$D_2 = \partial_\rho + \frac{2\lambda\rho}{\lambda^2 + \rho^2} \partial_\lambda = \partial_\rho|_{\lambda\,\text{fixed}} + \partial_\rho \lambda \partial_\lambda. \quad (3.16)$$

in the original formulation (3.6) ($\lambda$ was viewed there as an $x$-independent spectral parameter). The solution to equations (3.14) is

$$\lambda(\rho, z) = (w - z) \mp \sqrt{(z - w)^2 + \rho^2} \quad (3.17)$$

with $w$ an integration constant.

Similarly to the BM linear system that we discussed before, the BZ linear system is an equivalent way to pose the non-linear problem (3.4). The BZ solution generation process starts with a seed solution $M_{E,0}$ for which it is easy to deduce $\Psi_{E,0}(\lambda)$ such that $\Psi_{E,0}(\lambda = 0) = M_{E,0}$; that can be flat space, with $M_{E,0} = \mathbb{1}\,, \Psi_{E,0} = \mathbb{1}$. The steps of this inverse scattering method toward a new solution are :

$$M_{E,0} \to \Psi_{E,0} \xrightarrow{\text{dressing}} \chi(\lambda)\Psi_{E,0}(\lambda) = \Psi_E \xrightarrow{\lambda=0} M_E. \quad (3.18)$$

The main step in this process is to find the dressing matrix $\chi(\lambda)$. Substituting $\Psi_E = \chi(\lambda)\Psi_{E,0}(\lambda)$ in (3.11), yields the equations for $\chi$ :

$$D_1 \chi = \frac{\rho V - \lambda U}{\lambda^2 + \rho^2} \chi - \chi \frac{\rho V_0 - \lambda U_0}{\lambda^2 + \rho^2}, \quad D_2 \chi = \frac{\rho U + \lambda V}{\lambda^2 + \rho^2} \chi - \chi \frac{\rho U_0 + \lambda V_0}{\lambda^2 + \rho^2}. \quad (3.19)$$

Moreover, $\chi$ has to fulfill additional requirements, in order to ensure that the new solution $M_E$ is real and symmetric [9],[47],[49]. Note that symmetric translates to $\sharp$-invariant in the more general language of section 2.2; in the Ehlers coset $\mathrm{G}_E/\mathrm{K}_E = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, the $\sharp$ operation on matrices $\in \mathrm{SL}(2,\mathbb{R})$ is just transposition

$$g^\sharp = \tau(g^{-1}) = \eta \left( g^{T\,-1} \right)^{-1} \eta^{-1} = g^T \quad (3.20)$$

since $\eta = \mathrm{diag}(1,1)$ for $K_E = \mathrm{SO}(2)$. (For other symmetric spaces, the $\sharp$-operation can be more than plain transposition. This is why it is often referred to as "generalised transposition").

From (3.17), we have

$$w = -\frac{\rho^2}{2\lambda} + z + \frac{\lambda}{2}, \tag{3.21}$$

such that $w$ is invariant under the replacement $\lambda \to \frac{-\rho^2}{\lambda}$. In addition, we observe that

$$D_1 w = D_2 w = 0, \tag{3.22}$$

meaning that for any function of $w$ only, e.g. $C(w)$, we have [3]

$$D_1 C(w) = D_2 C(w) = 0. \tag{3.24}$$

We conclude from the above result that every solution $\Psi_E$ of (3.11) is determined up to a function $C(w)$ [11],[49]. Moreover, solutions of (3.11) have an additional property, namely if $\Psi_E$ is a solution such that $M_E = \Psi_E|_{\lambda=0}$ is symmetric, then $\Psi'_E$ defined as [49]

$$\Psi'_E = M_E (\Psi_E^T)^{-1} \left(\frac{-\rho^2}{\lambda}\right) \tag{3.25}$$

is also a solution of (3.11) and generally relates to $\Psi_E$ as

$$\Psi'_E = \Psi_E C(w), \tag{3.26}$$

for some function $C(w)$. Now taking the above relation, the definition (3.25) and that $\Psi_E(\lambda) = \chi(\lambda)\Psi_{E,0}(\lambda)$, we find that

$$\chi(\lambda)\Psi_{E,0}(\lambda) = \chi'(\lambda)\Psi_{E,0}(\lambda)C(w), \tag{3.27}$$

where

$$\chi'(\lambda) = M_E \chi^{T\,-1} \left(\frac{-\rho^2}{\lambda}\right) M_{E,0}^{-1} \tag{3.28}$$

satisfies equations (3.19). The requirement

$$\chi'(\lambda) = \chi(\lambda) \tag{3.29}$$

that Belinski and Zakharov have, ensures that $M_E$ is symmetric (the derivation of $\chi'$ relies on (3.25) with $M_E$ symmetric) but at the same time fixes the gauge freedom in (3.27).

[3]This is because

$$\begin{aligned} D_{1,2}C(w) &= \partial_{z,\rho}|_{\lambda\,\text{fixed}}\, C(w) + \partial_{z,\rho}\lambda\partial_\lambda C(w) = \\ &= \partial_w C(w)\partial_{z,\rho} w|_{\lambda\,\text{fixed}} + \partial_{z,\rho}\lambda\partial_w C(w)\partial_\lambda w = \\ &= \partial_w C(w) D_{1,2} w = 0. \end{aligned} \tag{3.23}$$

#### Relation of BZ and BM linear system - Ehlers formulation

At this point, it is clear that the two linear systems share common characteristics and of course (ultimately)generate solutions to the same problem. Both BM and BZ linear systems are differential equations for a generating function ($\mathcal{V}_E, \Psi_E$ respectively) depending on a spectral parameter and yielding the solution when this parameter goes to zero. It is natural to ask how these objects are related to each other. Starting from the spectral parameter and comparing (2.113) to (3.17) it is clear that [13],[47]

$$\lambda(\rho, z) = -\rho t(\rho, z). \tag{3.30}$$

Keeping the above relation in mind, the generating functions $\mathcal{V}_E(t,x), \Psi_E(\lambda, x)$ are related by

$$\Psi_E(\lambda, x) = V_E^{\sharp}(x)\mathcal{V}_E(t,x) \tag{3.31}$$

where the right hand side becomes $V_E^{\sharp}V_E = M_E$ at $t = 0$, consistent with (3.12). However, there are some issues that we must consider with regards to (3.31). In the BZ method, the generating function is dressed into a new one, through the matrix $\chi$ which generally has determinant different than one. This means that the new solution $M_E$ is no longer in the Ehlers group, therefore a non admissible solution. In order to obtain the "physical" matrix $M_E^{\text{phys}}$, one needs to rescale $M_E$ such that it fulfills the determinant condition. Moreover, the gauge fixing $\chi'(\lambda) = \chi(\lambda)$ of BZ is a strong condition which translates into fine tuning of integration constants in the BM approach. Recalling that the $\sharp$-operation on $t$-dependent matrices involves the replacement $t \to -\frac{1}{t}$, we may view relation (3.28) as as an analogous property. However, the gauge fixing of BZ does not occur in the BM approach. With all the above considerations, relation (3.31) should be thought of as a representative one and that the generating functions on either side of this relation are each a member of an equivalence class under gauge transformations; therefore, it is generally hard to match $\Psi_E$ and $\mathcal{V}_E$ that correspond to some solution through relation (3.31).

#### Relation of BZ and BM linear system - Matzner-Misner formulation

The relation of the generating functions $\Psi$ and $\mathcal{V}_{MM}$ proves to be trickier as a result of the factor $\rho$ in (3.9). We find that [44]

$$\Psi(\lambda, x) = \sqrt{2\rho t w}\, V_{MM}^T(x)\eta\mathcal{V}_{MM}(t,x). \tag{3.32}$$

Note that the generating function $\mathcal{V}_{MM}$ enters in the BM linear system when expressed in terms of Matzner-Misner data. In section 2.4, we only use the Ehlers language, which proves to be simpler and more convenient for calculations that follow. In principle however, one can state everything in the Matzner-Misner coset. We note that relation (3.32) is different to the one given in [13]. The latter maps the two generating functions at $\lambda = t = 0$ but not away from that point. The factor $\sqrt{2\rho t w}$ in (3.32) is needed to restore the mapping for all $\lambda, t$.

### Solitonic solutions

In general, solving for a dressing matrix $\chi$ to generate a solution from a given seed is a difficult problem. However, when $\chi$ has a special form, the problem becomes purely algebraic; this special form is the so-called "solitonic" ansatz [9],[10] :

$$\chi = \mathbb{1} + \sum_{k=1}^{N} \frac{R_k}{\lambda - \mu_k}, \tag{3.33}$$

i.e. $\chi$ is a function with simple poles at $\lambda = \mu_k$, $k = 1, 2, ..., N$. The pole positions $\mu_k$ and the matrices $R_k$ are functions of $\rho, z$. Putting $w = w_k$ in (3.17), with $w_k$, $k = 1, 2, ..., N$ distinct numbers that we will take here to be real, the two solutions for each pole trajectory $\mu_k$ read

$$\mu_k = -(z - w_k) \pm \sqrt{(z - w_k)^2 + \rho^2}. \tag{3.34}$$

The solutions with $+/-$ sign in front of the square root are referred to as solitons / antisolitons respectively. Now in order to deduce the form of $R_k$ we will look at the pole structure of the product $\chi^{-1}(\lambda)\chi(\lambda)$. At $\lambda = \mu_k$, we have that

$$\chi^{-1}(\mu_k) R_k = 0, \tag{3.35}$$

since $\chi^{-1}(\lambda)\chi(\lambda) = 1$ for all $\lambda$. This means that $\chi^{-1}(\mu_k)$, $R_k$ are degenerate and can be factorized as [11]

$$(R_k)_{ab} = n_a^{(k)} m_b^{(k)} \qquad \left(\chi^{-1}(\mu_k)\right)_{ab} = q_a^{(k)} p_b^{(k)} \tag{3.36}$$

where $n_a^{(k)}, m_b^{(k)}, q_a^{(k)}, p_b^{(k)}$ are vectors. In the BZ notation for the vectors, the lower index denotes the vector component while the upper index is the soliton index, enumerating the number of poles of the dressing matrix. Taking the equations (3.19) at $\lambda = \mu_k$ together with (3.35) and (3.36) we find the equations for the vectors $m_a^{(k)}$. The solutions can be expressed in terms of the seed solution $\Psi_{E,0}$ as follows

$$m_a^{(k)} = m_{0,b}^{(k)} \left[\Psi_{E,0}^{-1}(\mu_k, \rho, z)\right]_{ba}, \tag{3.37}$$

where $m_{0,b}^{(k)}$ are arbitrary constant vectors. We note that there is a freedom in rescaling the vectors $m_{0,b}^{(k)}$ by arbitrary constant factors. However, as will be made clear shortly, these overall factors do not change the final result of the solution generation process. Next, we need to determine the vectors $n_a^{(k)}$. We use the condition $\chi'(\lambda) = \chi(\lambda)$, where we substitute the ansatz (3.33) and put $\lambda = \frac{-\rho^2}{\mu_k}$, i.e. the poles of $\chi(-\rho^2/\lambda)$. The result is

$$n_a^{(k)} = \sum_{l=1}^{N} \mu_l^{-1} (\Gamma_{\mathrm{BZ}}^{-1})_{kl} m_c^{(l)} (M_{E,0})_{ca}, \tag{3.38}$$

where $\Gamma_{\rm BZ}$ is a symmetric matrix with elements

$$(\Gamma_{\rm BZ})_{kl} = \frac{m_a^{(k)}(M_{E,0})_{ab}m_b^{(l)}}{\rho^2 + \mu_k\mu_l}. \tag{3.39}$$

Combing the results for the vectors $m_a^{(k)}, n_a^{(k)}$ we find that the entries of the residue matrices $R_k$ read

$$(R_k)_{ab} = m_a^{(k)} \sum_{l=1}^{N} \frac{\left(\Gamma_{\rm BZ}^{-1}\right)_{lk} m_c^{(l)} \left(M_{E,0}\right)_{cb}}{\mu_l}. \tag{3.40}$$

Finally, the new solution $M_{E,0} = \Psi_E|_{\lambda=0} = \chi(0,\rho,z)\Psi_{E,0}(0,\rho,z)$ is given by

$$(M_E)_{ab} = (M_{E,0})_{ab} - \sum_{k,l=1}^{N} \frac{\left(M_{E,0}\right)_{ac} m_c^{(k)} \left(\Gamma_{\rm BZ}^{-1}\right)_{kl} m_d^{(l)} \left(M_{E,0}\right)_{db}}{\mu_k\mu_l}. \tag{3.41}$$

From the above expression and the formula (3.39), we can see that any arbitrary constants arising from the normalization freedom of $m_{0,b}^{(k)}$ will drop out. Indeed, these factors in the numerator of (3.41) will be cancelled out by the inverse scaling behaviour of matrix $\Gamma^{-1}$.

The new solution is guaranteed to be symmetric by the BZ requirement (3.29). However, its construction does not ensure that $M_E$ still has unit determinant, i.e. remains in the coset. Indeed, the determinant of $M_E$ is given by [9]

$$\det M_E = (-1)^N \rho^{2N} \left(\prod_{k=1}^{N} \mu_k^{-2}\right) \det M_{E,0}. \tag{3.42}$$

Therefore, in order to obtain a new solution that fulfills the requirements, we have to multiply $M_E$ with an appropriate factor. The formula for the physical solution can be generalised to $\mathrm{SL}(n,\mathbb{R})$ and it reads [4]

$$M_E^{(\rm phys)} = \pm \left(\frac{1}{\pm \det M_E}\right)^{\frac{1}{n}} M_E. \tag{3.43}$$

The choice of sign in front of $M_E$ is made in each case to ensure the right signature of the final metric.

At this point, we have solved for one part of the final metric (that is $M_E$) and need the conformal factor in front of the two-dimensional base metric to reach a complete new solution. Yet another nice feature of the solitonic case is that we can also reach the conformal factor algebraically. The calculation is very similar to the one in [9, 10] and the result for $\mathrm{SL}(2,\mathbb{R})$ reads

$$(f_E^{(\rm phys)})^2 = k_{\rm BZ} \cdot \rho^{N-\frac{N^2}{2}} \cdot \left(\prod_{k=1}^{N} \mu_k\right)^N \cdot \left[\prod_{k,l=1,\ k>l}^{N} (\mu_k - \mu_l)^2\right]^{-1} \cdot \det\Gamma_{\rm BZ} \cdot f_{E,0}^2, \tag{3.44}$$

[4]The formulae (3.42),(3.43) are adjusted for the Ehlers BZ method and differ slightly from the original ones for $\det g^{(\rm phys)}, g^{(\rm phys)}$ in [9, 10, 11].

with $k_{\mathrm{BZ}}$ an arbitrary numerical constant and $N$ the number of solitons.

For pure gravity in dimensions higher than four, i.e. in the case of $\mathrm{SL}(n,\mathbb{R})$ with $n > 2$, the fractional powers of $\rho$ in (3.43) coming from (3.42) typically lead to singular solutions (see [50] and references within). In five dimensions, this problem is overcome by the Pomeransky technique, where one starts with a suitable solitonic seed solution, from which solitons are "removed" and then "re-added" [46]. This is done in such a way that the generated solution is regular and satisfies the determinant condition. In this case, the conformal factor can be obtained through a general formula valid for $n \geq 2$ [46, 47].

## 3.2 Breitenlohner-Maison approach

In the Breitenlohner-Maison approach (cf. section 2.4), a new solution is generated via Geroch transformations of the monodromy matrix $\mathcal{M}_E(w)$. The most difficult part of this calculation is the factorization of $\mathcal{M}_E$ as

$$\mathcal{M}_E(w) = (\mathcal{V}_E(t,x))^\sharp\, \mathcal{V}_E(t,x). \tag{3.45}$$

In the soliton sector, where $\mathcal{M}_E(w)$ is a meromorphic function with simple poles in $w$, the Riemann-Hilbert problem (3.45) admits a solution that is obtained in a purely algebraic fashion. We start with the following ansatz for $\mathcal{M}_E(w)$

$$\mathcal{M}_E(w) = \mathbb{1} + \sum_{k=1}^{N} \frac{A_k}{w - w_k}, \tag{3.46}$$

that is a function with $N$ poles at $w = w_k$, $k = 1, 2, ..., N$. $A_k$ are constant residue matrices that we will take to be of rank one when $\mathrm{G}_E = \mathrm{SL}(2,\mathbb{R})_E \ni \mathcal{M}_E(w)$. Since $\det \mathcal{M}_E = 1$ and $\mathcal{M}_E(\infty) = \mathbb{1}$, we take the inverse matrix $\mathcal{M}_E(w)^{-1}$ to be a function with poles at the same points $w_k$ and residue matrices of rank one

$$\mathcal{M}_E(w)^{-1} = \mathbb{1} - \sum_{k=1}^{N} \frac{B_k}{w - w_k}. \tag{3.47}$$

For the coset space $\mathrm{G}_E/\mathrm{K}_E = \mathrm{SL}(2,\mathbb{R})_E/\mathrm{SO}(2)$, the $\sharp$-invariance of the monodromy matrix means that it is symmetric (see (3.20). Therefore, the residues $A_k, B_k$ must be symmetric too. We use a symmetric decomposition for the residue matrices using vectors $a_k, b_k$ as follows

$$A_k = \alpha_k a_k a_k^T, \qquad B_k = \beta_k b_k b_k^T, \tag{3.48}$$

where $\alpha_k, \beta_k$ are constant parameters. The ambiguity introduced by the choice of normalization for $a_k, b_k$ does not influence the final result of the calculation.

Similarly to the scaling freedom of the vectors $m_{0,b}^{(k)}$ in the BZ method, any constants multiplying the vectors $a_k, b_k$ disappear from the final solution [5].

We will start the factorization process from the following decomposition of the monodromy matrix $\mathcal{M}_E$:

$$\mathcal{M}_E(w) = A_-^T(t,x) M_E(x) A_+(t,x), \tag{3.49}$$

where we used the definition

$$\mathcal{V}_E(t,x) = V_E(x) A_+(t,x) \tag{3.50}$$

$$(\mathcal{V}_E(t,x))^\sharp = A_+^\sharp(-\frac{1}{t},x) V_E^\sharp(x) = A_-^T(t,x) V_E^T(x) \tag{3.51}$$

with $A_-(t,x) = A_+\left(-\frac{1}{t},x\right)$ and $\det A_+ = \det A_- = 1$. The matrix $M_E(x) = V_E^\sharp(x) V_E(x) = V_E(x)^T V_E(x)$ is symmetric and real. Using relation (2.113), we find that

$$\frac{1}{w - w_k} = \nu_k \left( \frac{t_k}{t - t_k} + \frac{1}{1 + t t_k} \right), \tag{3.52}$$

where $t_k$ are the moving poles

$$t_k = \frac{1}{\rho} \left( (z - w_k) + \sqrt{(z - w_k)^2 + \rho^2} \right), \tag{3.53}$$

given by (2.113) at $w = w_k$ with the plus sign in front of the square root. Moreover, $\nu_k$ are defined as

$$\nu_k = -\frac{2 t_k}{\rho \left(1 + t_k^2\right)}. \tag{3.54}$$

We use the above relations to express $\mathcal{M}_E(w)$ as a function of $(t,x)$. We find that

$$\mathcal{M}_E(t,x) = \mathbb{1} + \sum_{k=1}^{N} \frac{\nu_k t_k A_k}{t - t_k} + \sum_{k=1}^{N} \frac{\nu_k A_k}{1 + t t_k} \tag{3.55}$$

and

$$\mathcal{M}_E^{-1}(t,x) = \mathbb{1} - \sum_{k=1}^{N} \frac{\nu_k t_k B_k}{t - t_k} - \sum_{k=1}^{N} \frac{\nu_k B_k}{1 + t t_k}. \tag{3.56}$$

In this form, $\mathcal{M}_E$ has $2N$ poles, that is $N$ poles at $t = t_k$ and $N$ poles at $t = -\frac{1}{t_k}$. The pole structure of the product on the right hand side of (3.49) is such that the

[5] Similarly to the comment below (3.41), we can see that in the BM method too, the constant rescaling of the vectors $a_k, b_k$ leaves the final result unchanged. Indeed, the rescaling $a_k \to r_k a_k, b_k \to s_k b_k$ (by constants $r_k, s_k$) means that the parameters $\alpha_k, \beta_k$ must be changed to $r_k^{-2}\alpha_k, s_k^{-2}\beta_k$ to preserve the products (3.48). This in turn means that the entries $\Gamma_{kl}$ of the (BM) $\Gamma$ matrix will scale as $r_k s_l \Gamma_{kl}$ (see relations (3.61),(3.62),(3.65) that follow). With these considerations, the final result for $M_E$ given by (3.68) is clearly unaffected by these scalings.

poles at $t = t_k$ are due to $A_-(t,x)$ while the ones at $t = -\frac{1}{t_k}$ are due to $A_+(t,x)$ [13]. Thus we take $A_+$ to be of the form

$$A_+(t) = \mathbb{1} - \sum_{k=1}^{N} \frac{c_k t a_k^T}{1 + t t_k} \tag{3.57}$$

and its inverse

$$A_+^{-1}(t) = \mathbb{1} + \sum_{k=1}^{N} \frac{b_k t d_k^T}{1 + t t_k}, \tag{3.58}$$

where $c_k, d_k$ are new vectors to be determined later.

Using the expressions (3.55),(3.56) we can deduce conditions on the vectors $a_k, b_k$ by analysing the pole structure of the product $\mathcal{M}_E(t,x)\mathcal{M}_E(t,x)^{-1}$. First, the requirement that $\mathcal{M}_E\mathcal{M}_E^{-1}$ have no double poles at $t = -\frac{1}{t_k}$ yields the condition

$$a_k^T b_k = 0 \qquad \text{for each } k. \tag{3.59}$$

Next, we demand the absence of double poles in $\mathcal{M}_E\mathcal{M}_E^{-1}$ and arrive at the condition

$$a_k \alpha_k a_k^T \left[\mathcal{M}_E^{-1}(t,x) + \frac{b_k \nu_k \beta_k b_k^T}{1 + t t_k}\right]_{t=-\frac{1}{t_k}} = \left[\mathcal{M}_E(t,x) - \frac{a_k \nu_k \alpha_k a_k^T}{1 + t t_k}\right]_{t=-\frac{1}{t_k}} b_k \beta_k b_k^T, \tag{3.60}$$

which is fulfilled if there exist $\gamma_k$ such that

$$a_k^T \left[\mathcal{M}_E^{-1}(t,x) + \frac{b_k \nu_k \beta_k b_k^T}{1 + t t_k}\right]_{t=-\frac{1}{t_k}} = \gamma_k \nu_k \beta_k b_k^T \tag{3.61}$$

$$\left[\mathcal{M}_E(t,x) - \frac{a_k \nu_k \alpha_k a_k^T}{1 + t t_k}\right]_{t=-\frac{1}{t_k}} b_k = \gamma_k \alpha_k \nu_k a_k. \tag{3.62}$$

With the above relations, condition (3.60) holds, since both sides become equal to $\alpha_k \nu_k \beta_k \gamma_k (a_k b_k^T)$. The next step is to determine the vectors $c_k, d_k$. We have that the poles at $t = -\frac{1}{t_k}$ come from the matrix $A_+$ in the decomposition (3.49), therefore we consider the product $A_+\mathcal{M}_E^{-1}(t,x)$ and require that there are no poles at $t = -\frac{1}{t_k}$. The absence of double poles at $t = -\frac{1}{t_k}$ is ensured by the condition (3.59). The condition for no single poles, using the ansatz (3.57), becomes

$$t_k^{-1} c_k a_k^T \left[\mathcal{M}_E^{-1}(t,x) + \frac{b_k \nu_k \beta_k b_k^T}{1 + t t_k}\right]_{t=-\frac{1}{t_k}} = \left[\mathbb{1} - \sum_{i \neq k}^{N} \frac{t c_k a_k^T}{1 + t t_i}\right]_{t=-\frac{1}{t_k}} \nu_k B_k, \tag{3.63}$$

which is fulfilled when the following vector equation holds

$$c_k \Gamma_{kl} = b_l, \tag{3.64}$$

where $\Gamma$ is an $N \times N$ matrix with elements

$$\Gamma_{kl} = \begin{cases} \frac{\gamma_k}{t_k} & \text{for} \qquad k = l \\ \frac{1}{t_k - t_l} a_k^T b_l & \text{for} \qquad k \neq l. \end{cases} \tag{3.65}$$

and we have used the relations (3.61),(3.62). Defining $c, b$ as $2 \times N$ (or $n \times N$ for $G_E = \mathrm{SL}(n, \mathbb{R})$) matrices with columns the vectors $c_k, b_k$ respectively, we can write (3.64) as the matrix equation

$$c = b\Gamma^{-1}. \tag{3.66}$$

Similarly, from the requirement for no poles at $t = -\frac{1}{t_k}$ in the product $\mathcal{M}_E(t, x) A_+^{-1}$, we get the condition that determines the $d_k$ vectors [26]

$$d^T = \Gamma^{-1} a^T, \tag{3.67}$$

where the matrices $d, a$ are defined similarly to $c, b$ above. At this stage, we can reach the resulting formula for $M_E(x)$ in (3.49), using the fact that $\mathcal{M}_E(\infty) = \mathbb{1}$ and that $A_+(0, x) = \mathbb{1} = A_-(\infty, x)$ :

$$M_E = A_+^{-1}(\infty) = \mathbb{1} + \sum_{k,l=1}^{N} b_k t_k^{-1} (\Gamma^{-1})_{kl} a_l^T. \tag{3.68}$$

Once we obtain $M_E(x)$ we factorize it as $M_E = V_E^T V_E$ and read off the scalar fields from $V_E(x)$ (see parameterization (2.68) when $\mathrm{G}_E/\mathrm{K}_E = \mathrm{SL}(2, \mathbb{R})_E/\mathrm{SO}(2)$).

### Conformal factor

In the soliton sector, the conformal factor in the BM approach is given by a formula, as was the case in the BZ method (c.f. (3.44)). To reach this result [26], we start from the differential equations for the conformal factor $f_E$

$$\partial_\pm \ln \rho \, \partial_\pm \ln f_E = \frac{1}{2} \mathrm{Tr}(P_{E,\pm} P_{E,\pm}). \tag{3.69}$$

Next we write $\mathrm{Tr}(P_{E,\pm} P_{E,\pm})$ in terms of the matrix $A_+(t)$ . To achieve this, we evaluate the residue of the poles at $t = \pm i$ in the linear system (2.115). Using the relation

$$\partial_\pm \mathcal{V}_E(t, x) = \partial_\pm \mathcal{V}_E(t, x)|_t + (\partial_\pm t) \dot{\mathcal{V}}_E(t, x), \tag{3.70}$$

with $\dot{\mathcal{V}}_E(t, x) = \frac{\partial \mathcal{V}_E(t,x)}{\partial t}$, together with (2.110) gives

$$\pm \, i \partial_\pm \ln \rho \, \dot{\mathcal{V}}_E(\pm i) = P_{E,\pm} \mathcal{V}_E(\pm i). \tag{3.71}$$

Substituting $\mathcal{V}$ in (3.71) using relation (3.50), we obtain an expression for $P_{E,\pm}$ in terms of $A_+(\pm i)$ and $\dot{A}_+(\pm i)$. This expression is then used in the right hand side of (3.69) and gives

$$\partial_\pm \ln f_E = -\frac{1}{2} (\partial_\pm \ln \rho) \mathrm{Tr} \left( A_+^{-1}(\pm i) \dot{A}_+(\pm i) \right)^2. \tag{3.72}$$

Using the ansatz for $A_+(t)$ together with (3.66) and the identity $a_k^T b_k = \Gamma_{kl}(t_k - t_l)$ (see (3.65)) we get

$$A_+^{-1}(t)\dot{A}_+(t) = -b\frac{\mathbb{1}}{\mathbb{1}+tT}\Gamma^{-1}\frac{\mathbb{1}}{\mathbb{1}+tT}a^T, \tag{3.73}$$

where $T$ is an $N \times N$ diagonal matrix with entries $t_k$. Differentiating $a_k^T b_k = \Gamma_{kl}(t_k - t_l)$ with respect to the light cone coordinates for $k \neq l$, we obtain the components with $k \neq l$ of the equation

$$\partial_\pm \Gamma = -(\partial_\pm \ln \rho)\frac{\mathbb{1}}{\mathbb{1} \pm iT}\left[\Gamma \mp iT\Gamma \mp i\Gamma T + T\Gamma T\right]\frac{\mathbb{1}}{\mathbb{1} \pm iT}. \tag{3.74}$$

For the components with $k = l$ , we differentiate the relation (3.62). This calculation yields that (3.74) holds for the diagonal components $\Gamma_{kk}$ as well noting that the left hand side of (3.62) is constant. Now we use the result (3.73) into (3.72) and bring the right hand side to a convenient form involving terms that are total derivatives using (3.74). We find that 3.72 can be expressed in the form

$$\partial_\pm \ln f_E = \frac{1}{2}\mathrm{Tr}\left(\Gamma^{-1}\partial_\pm \Gamma\right) + \frac{1}{2}\mathrm{Tr}\left((T\nu)^{-1}\partial_\pm (T\nu)\right), \tag{3.75}$$

where $(T\nu)$ is the diagonal matrix with entries $t_k \nu_k$. Finally, integrating (3.75), we find

$$f_E^2 = k_{\mathrm{BM}} \cdot \prod_{k=1}^{N}(t_k \nu_k) \cdot \det \Gamma, \tag{3.76}$$

where $k_{\mathrm{BM}}$ is an integration constant that is determined according to desired properties for the final solution, such as asymptotic flatness.

### 3.2.1 Construction of the Kerr-NUT solution

It is known from calculations using the BZ method that (single-center) black hole solutions are 2-soliton solutions [11, 50]. Using the factorisation algorithm presented above, Breitenlohner and Maison have constructed the Schwarzschild and Kerr solutions in [26, 27][6]. We extend this study to the Kerr-NUT solution that we construct as the most general 2-soliton solution involving three parameters that correspond to mass, angular momentum and NUT charge.

We start with a general ansatz for $\mathcal{M}(w)$ with two poles

$$\mathcal{M}_E(w) = \mathbb{1} + \frac{\alpha_1 a_1 a_1^T}{w - c} + \frac{\alpha_2 a_2 a_2^T}{w + c}, \tag{3.77}$$

i.e. $N = 2$ and we have chosen, without loss of generality, the poles to be at $w_{1,2} = \pm c\,, c > 0$ . The vectors $a_1, a_2$ are two dimensional and $\alpha_1, \alpha_2$ are parameters

[6] In the class of solutions with two spacelike isometries, the construction of a colliding plane-wave solution in the BM approach is demonstrated in [31, 32].

that we will choose appropriately in what follows. We start with the vectors $a_1, a_2$ that we take to be of the form

$$a_1 = \begin{pmatrix} 1 \\ \zeta_1 \end{pmatrix}, \qquad a_2 = \begin{pmatrix} \zeta_2 \\ 1 \end{pmatrix}, \tag{3.78}$$

making use of the normalization freedom that was discussed earlier to set one of the components to one. Defining the matrix $\xi$ as

$$\xi = a^T a \quad \text{with } a = (a_1\ a_2) = \begin{pmatrix} 1 & \zeta_2 \\ \zeta_1 & 1 \end{pmatrix}, \tag{3.79}$$

we take [26]

$$\alpha = \frac{2c}{\det \xi} \begin{pmatrix} \xi_{22} & 0 \\ 0 & -\xi_{11} \end{pmatrix}, \qquad \alpha = \mathrm{diag}\{\alpha_1, \alpha_2\}, \tag{3.80}$$

such that the condition $\det \mathcal{M}(w) = 1$ is fulfilled for the ansatz (3.77). Next, we can choose the matrices $b = (b_1\, b_2), \beta = \mathrm{diag}\{\beta_1,\, \beta_2\}$ as

$$b = a\xi^{-1}\epsilon\,, \qquad \beta = -\alpha \det\xi \tag{3.81}$$

with

$$\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.82}$$

Therefore we get

$$\xi = \begin{pmatrix} 1+\zeta_1^2 & \zeta_1+\zeta_2 \\ \zeta_1+\zeta_2 & 1+\zeta_2^2 \end{pmatrix} \tag{3.83}$$

and for the parameters $\alpha_1, \alpha_2$

$$\alpha_1 = \frac{2c(1+\zeta_2^2)}{(1-\zeta_1\zeta_2)^2} \qquad \alpha_2 = -\frac{2c(1+\zeta_1^2)}{(1-\zeta_1\zeta_2)^2}. \tag{3.84}$$

The $b_1, b_2$ vectors read

$$b_1 = \frac{1}{1-\zeta_1\zeta_2} \begin{pmatrix} -\zeta_1 \\ 1 \end{pmatrix}, \qquad b_2 = \frac{1}{1-\zeta_1\zeta_2} \begin{pmatrix} -1 \\ \zeta_2 \end{pmatrix}, \tag{3.85}$$

and the parameters $\beta_1, \beta_2$

$$\beta_1 = -2c(1+\zeta_2^2), \qquad \beta_2 = 2c(1+\zeta_1^2). \tag{3.86}$$

For the vectors $a_1, a_2, b_1, b_2$ the following relations hold

$$a_k^T b_k = 0 \ \text{ for } k = 1,2 \quad \text{and} \quad a_2^T b_1 = -a_1^T b_2 = 1. \tag{3.87}$$

Next, from equations (3.61),(3.62) we find the $\gamma_{1,2}$

$$\gamma_1 = \frac{t_2(1+t_1^2)(\zeta_1+\zeta_2)}{(t_2-t_1-t_1^2t_2+t_1t_2^2)(1+\zeta_2^2)} \tag{3.88}$$

$$\gamma_2 = \frac{t_1(1+t_2^2)(\zeta_1+\zeta_2)}{(t_2-t_1-t_1^2t_2+t_1t_2^2)(1+\zeta_1^2)} \tag{3.89}$$

and together with (3.65),(3.87) we get the $\Gamma$ matrix

$$\Gamma = \frac{1}{t_2-t_1}\begin{pmatrix} \frac{\xi_{12}}{\xi_{22}}\frac{t_2(t_1+t_1^{-1})}{1+t_1t_2} & 1 \\ 1 & \frac{\xi_{12}}{\xi_{11}}\frac{t_1(t_2+t_2^{-1})}{1+t_1t_2} \end{pmatrix}. \tag{3.90}$$

The next step is to find the $c_{1,2}$ vectors from (3.66) (or the $d_{1,2}$ vectors from (3.67)). Finally, we reach the matrix $M(x)$ as $M = A_+^{-1}(t)|_{t\to\infty}$. From (3.68) and the choice (3.81) we have that [26]

$$M_E(x) = \mathbb{1} + a(\Gamma T\epsilon^{-1}\xi)^{-1}a^T, \qquad \text{where} \qquad T = \mathrm{diag}\,\{t_1, t_2\}. \tag{3.91}$$

For the conformal factor, we use the formula (3.76) and find

$$f_E^2 = k_{\mathrm{BM}}\frac{t_1\nu_1t_2\nu_2}{(t_2-t_1)^2}\left[\frac{\xi_{12}^2}{\xi_{11}\xi_{22}}\frac{(1+t_1^2)(1+t_2^2)}{(1+t_1t_2)^2} - 1\right]. \tag{3.92}$$

**Interpretation as Kerr-NUT metric**

From the parameterization (2.2) and (2.11), we see that the factor in front of the base metric is equal to $\Delta^{-1}{f_E}^2$ while according to section (3.1), the Killing part $g$ of the metric is related to the Matzner-Misner coset metric as $g = \rho\bar{\eta}M_{MM}$. Therefore, in order to write the final spacetime metric, we need to first read off the scalars in the Ehlers coset from the solution $M_E$ that we obtained in the previous section and then integrate the duality relation (2.20) to get the Matzner-Misner scalar $\psi$. The four-dimensional line element has the form

$$ds_4^2 = -\Delta\,(dt+\psi d\phi)^2 + \Delta^{-1}(f_E^2\left(d\rho^2+dz^2\right)+\rho^2d\phi^2). \tag{3.93}$$

It is more convenient to use prolate coordinates $(u,v)$ to present the solutions for the scalar fields. The relation of $(u,v)$ to the Weyl coordinates is

$$z = uv, \qquad \rho = \sqrt{(u^2-c^2)(1-v^2)}, \qquad c \le u < \infty, \qquad -1 \le v \le 1. \tag{3.94}$$

The pole trajectories $t_1$ and $t_2$ as functions of $u, v$ read

$$t_1 = \frac{(u-c)(1+v)}{\sqrt{(u^2-c^2)(1-v^2)}}, \qquad t_2 = \frac{(u+c)(1+v)}{\sqrt{(u^2-c^2)(1-v^2)}}. \tag{3.95}$$

Moreover,

$$\frac{t_1}{t_2} = \frac{u-c}{u+c}, \qquad t_1t_2 = \frac{1+v}{1-v}. \tag{3.96}$$

Using the above relations, we get the following expressions for the scalar fields $\Delta, \tilde{\psi}$ [44] [7]

$$\Delta = \frac{1}{D}\left[v^2c^2(\zeta_1+\zeta_2)^2 + u^2(1-\zeta_1\zeta_2)^2 - c^2(1+\zeta_1^2)(1+\zeta_2^2)\right] \tag{3.97}$$

and

$$\tilde{\psi} = \frac{1}{D}\left[2cu(\zeta_2-\zeta_1)(1-\zeta_1\zeta_2) - 2c^2v(\zeta_1+\zeta_2)(1+\zeta_1\zeta_2)\right], \tag{3.98}$$

where the denominator $D$ in these expressions is

$$D = v^2c^2(\zeta_1+\zeta_2)^2 + 2vc^2(\zeta_2^2-\zeta_1^2) + u^2(1-\zeta_1\zeta_2)^2 + c^2(1+\zeta_1^2)(1+\zeta_2^2) + 2cu(1-\zeta_1^2\zeta_2^2). \tag{3.99}$$

Integrating (2.20) one can write an expression for $\psi$.[8] It is slightly more complicated and reads

$$\psi = \frac{N_\psi}{(1-\zeta_1\zeta_2)D_\psi}, \tag{3.100a}$$

$$D_\psi = u^2(1-\zeta_1\zeta_2)^2 - c^2(1+\zeta_1^2)(1+\zeta_2^2) - c^2v^2(\zeta_1+\zeta_2)^2, \tag{3.100b}$$

$$\begin{aligned} N_\psi = &-4c^3\zeta_1(1+\zeta_1^2)(1+\zeta_2^2) - 2c^2(\zeta_1+\zeta_2)(1-\zeta_1^2\zeta_2^2)u + 2c(\zeta_1-\zeta_2) \\ &(1-\zeta_1\zeta_2)^2u^2 - 2c^3(\zeta_1-\zeta_2)(1-\zeta_1\zeta_2)^2v + 2c(\zeta_1-\zeta_2)(1-\zeta_1\zeta_2)^2u^2v \\ &+ 2c^3(\zeta_1+\zeta_2)(1+2\zeta_1^2+\zeta_1^2\zeta_2^2)v^2 + 2c^2(\zeta_1+\zeta_2)(1-\zeta_1^2\zeta_2^2)uv^2. \end{aligned} \tag{3.100c}$$

At this point, we have not yet established the relation of the parameters $c, \zeta_1, \zeta_2$ to the familiar parameters of the Kerr-NUT solution, i.e. the mass $m$, the rotational parameter $a$ and the NUT parameter $n$. In order to do so, we compare the metric components of (3.93) with the above solutions for the scalars $\Delta, \psi$ to the Kerr-NUT metric

$$\begin{aligned} ds^2 = &-\frac{1}{\Sigma}\left(\Xi - a^2\sin^2\theta\right)dt^2 + \frac{\Sigma}{\Xi}dr^2 + \Sigma d\theta^2 + \frac{1}{\Sigma}\left((\Sigma+a\chi)^2\sin^2\theta - \chi^2\Xi\right)d\phi^2 \\ &+ \frac{2}{\Sigma}\left(\chi\Xi - a(\Sigma+a\chi)\sin^2\theta\right)dtd\phi, \end{aligned} \tag{3.101}$$

with

$$\Sigma = r^2 + (n + a\cos\theta)^2 \tag{3.102a}$$

$$\Xi = r^2 - 2mr - n^2 + a^2 \tag{3.102b}$$

$$\chi = a\sin^2\theta - 2n(1+\cos\theta) \tag{3.102c}$$

[7] This calculation was performed following the conventions in [13] where the entries in $(g_{ab}) = \rho\bar{\eta}M_{MM}$ are ordered as $\begin{pmatrix} g_{\phi\phi} & g_{\phi t} \\ g_{t\phi} & g_{tt} \end{pmatrix}$. In the parameterization (2.43),(2.61) this corresponds to exchanging the diagonal entries.

[8] Alternatively, one could construct the generating function $\mathcal{V}_E(t)$ and apply the algebraic Kramer–Neugebauer transformation (relating the Ehlers and Matzner–Misner description) to obtain $\mathcal{V}_{MM}(t)$ that directly contains $\psi$ [13].

and use the relation of the $u, v$ coordinates to the standard Boyer-Lindquist coordinates $(r, \theta)$ :

$$u = r - m, \qquad v = \cos\theta. \tag{3.103}$$

Identifying the parameters $\zeta_1, \zeta_2$ as follows

$$\zeta_1 = \frac{c-m}{a+n}, \qquad \zeta_2 = -\frac{a+n}{c+m}, \qquad c = \sqrt{m^2+n^2-a^2}, \tag{3.104}$$

we find that the metric (3.93) matches the Kerr-NUT metric (3.101). From (3.104) we find

$$a = -m\frac{\zeta_1+\zeta_2}{1+\zeta_1\zeta_2}, \qquad n = m\frac{\zeta_1-\zeta_2}{1+\zeta_1\zeta_2}, \qquad m = c\frac{1+\zeta_1\zeta_2}{1-\zeta_1\zeta_2}. \tag{3.105}$$

When $\zeta_1 = \zeta_2$, we recover the Kerr solution. The expressions for the scalars agree with the ones in [27] and [26] (once certain typos are corrected). The Schwarzschild solution is recovered when $\zeta_1 = \zeta_2 = 0$ [27, 32].

Finally we can compute the conformal factor for the Kerr-NUT solution using the formula (3.76). We find

$$f_E^2 = -k_{\rm BM}\frac{u^2-m^2-n^2+a^2v^2}{4(m^2+n^2)(u^2-c^2v^2)}. \tag{3.106}$$

We choose the constant $k_{\rm BM}$ such that $f_{\rm E} \to 1$ as $r \to \infty$, which gives $k_{\rm BM} = -4(m^2+n^2)$. We have

$$f_E^2 = \frac{u^2-m^2-n^2+a^2v^2}{(u^2-c^2v^2)} \tag{3.107}$$

and in the final spacetime metric the conformal factor reads

$$\Delta^{-1}f_E^2 = \frac{(m+u)^2+(n+av)^2}{u^2-c^2v^2}. \tag{3.108}$$

which agrees with (3.101). Using the relations (3.104) and the ansatz (3.77), the monodromy matrix $\mathcal{M}_E(w)$ that corresponds to the Kerr-NUT solution takes the form

$$\mathcal{M}_E(w) = \frac{1}{w^2-c^2}\begin{pmatrix} (m+w)^2+(n+a)^2 & 2(am-nw) \\ 2(am-nw) & (w-m)^2+(a-n)^2 \end{pmatrix} \tag{3.109}$$

with $c$ as in (3.104). (Setting $n=0, a=0$, we get the monodromy matrix of the Schwarzschild solution given in [27].)

# Chapter 4

# Integrability in STU supergravity

In this chapter, we will present the generalization and application of the BM technique to STU supergravity. This is a gravity theory, that upon reduction to three dimensions, belongs in the class of $\mathrm{G_E/K_E}$ $\sigma$-models discussed in the first chapter. The global Ehlers group $G_E$ is $\mathrm{SO}(4,4)$ and (for the sequence of reductions discussed here) the denominator group $K_E$ is $\mathrm{SO}(2,2) \times \mathrm{SO}(2,2)$. In the joint work with A. Kleinschmidt and A. Virmani [51], we study the case of four-dimensional asymptotically flat solutions within the STU model. We provide a generalisation of the BM method for solitonic solutions that is adjusted to the group structure in this model and exhibit its applicability by reconstructing explicitly a (non-extremal), charged, rotating black hole solution. The most general black hole solution in this set-up involves eight charges, four electric and four magnetic [52, 53]. For our purposes, which are mainly to test the inverse scattering approach to STU supergravity, we will aim for the Cvetic-Youm [54, 28] four-charge solution with one electric and three magnetic charges [28, 55, 56, 57, 58]. As we will see shortly, a new element in the generalisation of the BM algorithm to $\mathrm{SO}(4,4)/(\mathrm{SO}(2,2)\times\mathrm{SO}(2,2))$ is the increased rank of the residue matrices in the solitonic ansatz (rank is two as opposed to one in Einstein gravity). This development motivated us to phrase the formalism in a general way, accommodating all cases with rank $r \geq 1$. This extension could cover other coset models with different groups involved, assuming that it is possible to work with the matrix representation of the respective group elements.

## 4.1 STU model and preliminaries

Let us start with a brief description of the four-dimensional STU supergravity model. The latter is $\mathcal{N} = 2$ supergravity coupled to three vector multiplets and the letters "S-T-U" refer to the names of the three complex scalars in the theory, namely the dilaton/axion, the Kähler form field and the structure field respectively. In our work, we choose to start from ten-dimensional type IIB string theory (other string theories that are related by duality can be taken as starting points; these different duality frames correspond to interchanges of the S,T,U fields [59, 60]) and obtain the STU model through a series of dimensional reductions. First reducing on a four-torus, we obtain a six-dimensional consistent truncation of the parent theory and with subsequent reduction on a two-torus, obtain the so-called STU model in four dimensions. Another route to this model is the reduction of M-theory on a six-torus, resulting in the five-dimensional $U(1)^3$ supergravity theory which in turn gives four-dimensional STU supergravity upon further reduction. We will be brief in our preliminary discussion on the STU model, since the details are well documented in the literature [59, 60, 56]. For the most part, we will follow the notation and conventions of [61, 58].

Let us consider a consistent truncation of the ten-dimensional string theory on $T^4$. We write the line element as

$$ds_{10}^2 = ds_6^2 + e^{\frac{\Phi}{\sqrt{2}}} ds_4^2, \tag{4.1}$$

where $\Phi = \sqrt{2}\Phi_{10}$ with $\Phi_{10}$ the type IIB dilaton. In the six-dimensional effective theory, the bosonic part is described by the Lagrangian [60]

$$\mathcal{L}^{(6d)} = R^{(6d)} \star_6 1 - \frac{1}{2} \star_6 d\Phi \wedge d\Phi - \frac{1}{2} e^{\sqrt{2}\Phi} \star_6 F_{[3]} \wedge F_{[3]}, \tag{4.2}$$

where the three-form field $F_{[3]}$ is given by $F_{[3]} = dC_{[2]}$, with $C_{[2]}$ the Ramond-Ramond two-form of the IIB theory.

For the calculations in this chapter, the dimensional reduction from six to three dimensions will be carried out in the following way

$$D = 6 \xrightarrow[z_6]{\text{spacelike}} D = 5 \xrightarrow[z_5]{\text{spacelike}} D = 4 \xrightarrow[t]{\text{timelike}} D = 3. \tag{4.3}$$

Thus, starting from reduction of (4.2) on a circle, with a standard Kaluza-Klein ansatz, the metric can be written in the form

$$ds_6^2 = e^{-\sqrt{\frac{3}{2}}\phi_6} \left( dz_6 + A^1_{[1]} \right)^2 + e^{\sqrt{\frac{1}{6}}\phi_6} ds_5^2 \tag{4.4}$$

and the six-dimensional three-form $F_{[3]}$ can be written as

$$F_{[3]} = F^{(5d)}_{[3]} + dA^2_{[1]} \wedge \left( dz + A^1_{[1]} \right) \quad \text{with} \quad F^{(5d)}_{[3]} = dC^{(5d)}_{[2]} - dA^2_{[1]} \wedge dA^1_{[1]}. \tag{4.5}$$

The Lagrangian in five dimensions is given by

$$\begin{aligned}\mathcal{L}^{(5d)} = {} & R^{(5d)} \star_5 1 - \frac{1}{2} \star_5 d\Phi \wedge d\Phi - \frac{1}{2} \star_5 d\phi_6 \wedge d\phi_6 - \frac{1}{2} e^{-2\sqrt{\frac{2}{3}}\phi_6} \star_5 F^1_{[2]} \wedge F^1_{[2]} \\ & - \frac{1}{2} e^{-\sqrt{\frac{2}{3}}\phi_6 + \sqrt{2}\Phi} \star_5 F^{(5d)}_{[3]} \wedge F^{(5d)}_{[3]} - \frac{1}{2} e^{\sqrt{\frac{2}{3}}\phi_6 + \sqrt{2}\Phi} \star_5 F^2_{[2]} \wedge F^2_{[2]}\end{aligned} \tag{4.6}$$

with $F^1_{[2]}, F^2_{[2]}$ the two-forms given by $F^1_{[2]} = dA^1_{[1]}, F^2_{[2]} = dA^2_{[1]}$. The U$(1)^3$ structure of this theory arises when the two-form that is "hidden" in $F^{(5d)}_{[3]}$ is dualized into a new one-form $A^3_{[1]}$. A similar process was followed in section 2.1.1, where a two-form was dualized into a scalar, thereby leading to the Ehlers Lagrangian of Einstein gravity in three-dimensions. The same strategy applies here, but in five dimensions, duality relates a two-form to an one-form. (More generally, in $D$-dimensions the dual object to a $p$-form is $(D-p-2)$-dimensional). We add a term to the Lagrangian which vanishes by virtue of the Bianchi identity for the field $F^{(5d)}_{[3]}$ and vary the new Lagrangian with respect to the same field. The equation for $F^{(5d)}_{[3]}$ reads

$$F^{(5d)}_{[3]} = -e^{\sqrt{\frac{2}{3}}\phi_6 - \sqrt{2}\Phi} \star_5 F^3_{[2]} \tag{4.7}$$

where $F^3_{[2]} = dA^3_{[1]}$. Substituting $F^{(5d)}_{[3]}$ from the above equation into the new Lagrangian and changing the parameterization for the scalars

$$h^1 = e^{\sqrt{\frac{2}{3}}\phi_6}, \quad h^2 = e^{-\sqrt{\frac{1}{6}}\phi_6 - \sqrt{\frac{1}{2}}\Phi}, \quad h^3 = e^{-\sqrt{\frac{1}{6}}\phi_6 + \sqrt{\frac{1}{2}}\Phi}, \tag{4.8}$$

such that $h^1h^2h^3 = 1$, we obtain the Lagrangian

$$\mathcal{L}^{(5d)} = R^{(5d)} \star_5 1 - \frac{1}{2} G_{IJ} \star_5 dh^I \wedge dh^J - \frac{1}{2} G_{IJ} \star_5 F^I_{[2]} \wedge F^J_{[2]} + \frac{1}{6} C_{IJK} F^I_{[2]} \wedge F^J_{[2]} \wedge A^K_{[1]}. \tag{4.9}$$

The index $I$ takes values $I = 1, 2, 3$, $(G_{IJ})$ is diagonal with entries $G_{II} = (h^I)^{-2}$ and the symbol $C_{IJK}$ is symmetric under the interchange of pairs of indices with $C_{123} = 1$.

Let us now reduce this theory to four dimensions. Our aim is to reach a Lagrangian that is equivalent to the (bosonic part of the) STU Lagrangian with the appropriate field identifications. We decompose the five-dimensional metric as follows

$$ds_5^2 = \check{f}^2 \left(dz_5 + \check{A}^0_{[1]}\right)^2 + \check{f}^{-1} ds_4^2 \tag{4.10}$$

and the five-dimensional vectors $A^I_{[1]}$ as

$$A^I_{[1]} = \chi^I \left(dz_5 + \check{A}^0_{[1]}\right) + \check{A}^I_{[1]}. \tag{4.11}$$

Defining the two-form field strengths $\check{F}^\Lambda_{[2]} = d\check{A}^\Lambda_{[1]}$ with $\Lambda = 0, 1, 2, 3$, the reduction of (4.9) to four dimensions reads

$$\begin{aligned}\mathcal{L}^{(4d)} = {} & R^{(4d)} \star_4 1 - \frac{1}{2} G_{IJ} \star_4 dh^I \wedge dh^J - \frac{3}{2} \check{f}^{-2} \star_4 d\check{f} \wedge d\check{f} - \frac{1}{2} \check{f}^3 \star_4 \check{F}^0_{[2]} \wedge \check{F}^0_{[2]} \\ & - \frac{1}{2} \check{f}^{-2} G_{IJ} \star_4 d\chi^I \wedge d\chi^J - \frac{1}{2} \check{f} G_{IJ} \star_4 \left(\check{F}^I_{[2]} + \chi^I \check{F}^0_{[2]}\right) \wedge \left(\check{F}^J_{[2]} + \chi^J \check{F}^0_{[2]}\right) \\ & + \frac{1}{2} C_{IJK} \chi^I \check{F}^J_{[2]} \wedge \check{F}^K_{[2]} + \frac{1}{2} C_{IJK} \chi^I \chi^J \check{F}^0_{[2]} \wedge \check{F}^K_{[2]} + \frac{1}{6} C_{IJK} \chi^I \chi^J \chi^K \check{F}^0_{[2]} \wedge \check{F}^0_{[2]}.\end{aligned} \tag{4.12}$$

Now if we combine $\chi^I$ and $h^I$ into complex scalar fields $z^I$ such that $z^I = -\chi^I + i\check{f}h^I$ and make the identification $z^I = \frac{X^I}{X^0}$, we can match (4.12) to the STU theory with prepotential

$$F = -\frac{X^1 X^2 X^3}{X^0} \tag{4.13}$$

in the gauge $X^0 = 1$. The latter theory can be written in the form [62]

$$\mathcal{L}^{(4d)} = R^{(4d)} \star_4 1 - 2g_{I\bar{J}} \star_4 dX^I d\bar{X}^{\bar{J}} + \frac{1}{2}\check{F}^{\Lambda}_{[2]} \wedge \check{G}_{\Lambda[2]}. \tag{4.14}$$

The metric $g_{I\bar{J}}$ is the Kähler metric given by $g_{I\bar{J}} = \partial_I \partial_{\bar{J}} K$, with Kähler potential $K = -\log\left(-i\left(\bar{X}^{\Lambda} F_{\Lambda} - \bar{F}_{\Lambda} X^{\Lambda}\right)\right)$. With the definition of the complex symmetric matrix $N_{\Lambda\Sigma}$

$$N_{\Lambda\Sigma} = \bar{F}_{\Lambda\Sigma} + 2i \frac{\mathrm{Im}F_{\Lambda K}\, \mathrm{Im}F_{\Sigma P}\, X^K X^P}{\mathrm{Im}F_{MN}\, X^M X^N}, \tag{4.15}$$

where $F_{\Lambda} = \partial_{\Lambda} F$ and $F_{\Lambda\Sigma} = \partial_{\Lambda}\partial_{\Sigma} F$, the two-forms $\check{G}_{\Lambda[2]}$ are given by

$$\check{G}_{\Lambda[2]} = (\mathrm{Re}N)_{\Lambda\Sigma}\, \check{F}^{\Sigma}_{[2]} + (\mathrm{Im}N)_{\Lambda\Sigma} \star_4 \check{F}^{\Sigma}_{[2]}. \tag{4.16}$$

In our set-up, where $z^I = \frac{X^I}{X^0} = X^I = -\chi^I + i\check{f}h^I$ and setting $x^I = -\chi^I, y^I = \check{f}h^I$, we have that

$$z^I = x^I + iy^I, \tag{4.17}$$

such that we write for $N_{\Lambda\Sigma}$

$$(\mathrm{Re}N)_{\Lambda\Sigma} = \begin{pmatrix} -2x_1x_2x_3 & x_2x_3 & x_1x_3 & x_1x_2 \\ x_2x_3 & 0 & -x_3 & -x_2 \\ x_1x_3 & -x_3 & 0 & -x_1 \\ x_1x_2 & -x_2 & -x_1 & 0 \end{pmatrix}, \tag{4.18}$$

and

$$(\mathrm{Im}N)_{\Lambda\Sigma} = \begin{pmatrix} \frac{-x_3^2y_1^2y_2^2 - x_1^2y_3^2y_2^2 - x_2^2y_1^2y_3^2 - y_1^2y_2^2y_3^2}{y_1y_2y_3} & \frac{x_1y_2y_3}{y_1} & \frac{x_2y_1y_3}{y_2} & \frac{x_3y_1y_2}{y_3} \\ \frac{x_1y_2y_3}{y_1} & -\frac{y_2y_3}{y_1} & 0 & 0 \\ \frac{x_2y_1y_3}{y_2} & 0 & -\frac{y_1y_3}{y_2} & 0 \\ \frac{x_3y_1y_2}{y_3} & 0 & 0 & -\frac{y_1y_2}{y_3} \end{pmatrix}, \tag{4.19}$$

with inverse

$$((\mathrm{Im}N)^{-1})_{\Lambda\Sigma} = \frac{1}{y_1y_2y_3} \begin{pmatrix} -1 & -x_1 & -x_2 & -x_3 \\ -x_1 & -x_1^2 - y_1^2 & -x_1x_2 & -x_1x_3 \\ -x_2 & -x_1x_2 & -x_2^2 - y_2^2 & -x_2x_3 \\ -x_3 & -x_1x_3 & -x_2x_3 & -x_3^2 - y_3^2 \end{pmatrix}, \tag{4.20}$$

where we have lowered the index $I$ to make the notation clearer.

## Reduction to D=3: the $\mathrm{SO}(4,4)/(\mathrm{SO}(2,2)\times\mathrm{SO}(2,2))$ coset model

Further reduction of (4.12), along a *timelike* direction, leads to a gravity-matter system that is described by a $\mathrm{G}_E/\mathrm{K}_E = \mathrm{SO}(4,4)/(\mathrm{SO}(2,2)\times\mathrm{SO}(2,2))$ coset model. The matter part is purely scalar and is parameterized by $\dim(\mathrm{SO}(4,4)/(\mathrm{SO}(2,2)\times\mathrm{SO}(2,2)))$ scalar fields, i.e. $\dim(\mathrm{SO}(4,4)) - \dim(\mathrm{SO}(2,2)\times\mathrm{SO}(2,2)) = 28-12 = 16$ fields. Indeed, there are six scalars $(x^I, y^I)$ that are already present in four dimensions while another five scalars originate from the Kaluza-Klein reduction; the remaining five come from the dualization of three-dimensional vectors. In more detail, starting from the parameterization of the metric as

$$ds_4^2 = -e^{2U}(dt+\omega_3)^2 + e^{-2U}ds_3^2 \quad \text{with} \quad ds_3^2 = f_E^2(d\rho^2+dz^2)+\rho^2 d\phi^2 \tag{4.21}$$

and the four-dimensional vectors $\check{A}^{\Lambda}_{[1]}$, $\Lambda = 0,1,2,3$ that we write as

$$\check{A}^{\Lambda}_{[1]} = \zeta^{\Lambda}(dt+\omega_3) + A^{\Lambda}_3, \tag{4.22}$$

we get the scalars $U, \zeta^{\Lambda}$ and the one-forms $\omega_3, A^{\Lambda}_3$. The latter are dualized into scalars $(\sigma, \tilde{\zeta}_{\Lambda})$ according to the relations

$$d\sigma - \frac{1}{2}\left(\zeta^{\Lambda} d\tilde{\zeta}_{\Lambda} - \tilde{\zeta}_{\Lambda} d\zeta^{\Lambda}\right) = -e^{4U} \star d\omega_3 \tag{4.23}$$

and

$$-d\tilde{\zeta}_{\Lambda} = e^{2U}(\mathrm{Im}N)_{\Lambda\Sigma} \star \left(dA^{\Sigma}_3 + \zeta^{\Sigma} d\omega_3\right) + (\mathrm{Re}N)_{\Lambda\Sigma} d\zeta^{\Sigma}. \tag{4.24}$$

Thus we have a set of 16 scalars in three-dimensions that we denote as $\varphi^i = \left(U, z^I, \bar{z}^I, \zeta^{\Lambda}, \tilde{\zeta}_{\Lambda}, \sigma\right)$ and the Lagrangian takes the $\sigma$-model form (cf. (2.21))

$$\mathcal{L}^{(3d)} = R^{(3d)} \star_3 1 - \frac{1}{2} h_{ij} \partial\varphi^i \partial\varphi^j, \tag{4.25}$$

where the metric $h_{ij}$ on the manifold $\mathrm{G}_E/\mathrm{K}_E = \mathrm{SO}(4,4)/(\mathrm{SO}(2,2)\times\mathrm{SO}(2,2))$ reads

$$\begin{aligned} h_{ij} d\varphi^i d\varphi^j =& 4dU^2 + 4g_{I\bar{J}} dz^I d\bar{z}^{\bar{J}} + \frac{1}{4} e^{-4U}\left(d\sigma + \tilde{\zeta}_{\Lambda} d\zeta^{\Lambda} - \zeta^{\Lambda} d\tilde{\zeta}_{\Lambda}\right)^2 \\ &+ e^{-2U}\Big((\mathrm{Im}N)_{\Lambda\Sigma} d\zeta^{\Lambda} d\zeta^{\Sigma} + ((\mathrm{Im}N)^{-1})^{\Lambda\Sigma}\left(d\tilde{\zeta}_{\Lambda} + (\mathrm{Re}N)_{\Lambda P} d\zeta^P\right) \\ &\left(d\tilde{\zeta}_{\Sigma} + (\mathrm{Re}N)_{\Sigma P} d\zeta^P\right)\Big). \end{aligned} \tag{4.26}$$

Following the discussion in section 2.2 we write (4.25) as

$$\mathcal{L}^{(3d)} = \sqrt{g_3}\left(R^{(3d)} - \frac{1}{2} g^{mn} \mathrm{Tr}(P_m P_n)\right), \tag{4.27}$$

that is

$$g^{mn}\partial_m\varphi^i\partial_n\varphi^j h_{ij} = g^{mn}\mathrm{Tr}(P_m P_n), \tag{4.28}$$

where $P_m$ is given by (2.32b). The generalised transposition $\sharp$ is the anti-involution that fixes the subgroup $\mathrm{SO}(2,2)\times\mathrm{SO}(2,2)$ through the relation

$$k^\sharp k = 1, \quad k \in \mathrm{SO}(2,2)\times\mathrm{SO}(2,2) \tag{4.29}$$

and acts on the group elements $g \in \mathrm{SO}(4,4)$ as

$$g^\sharp = \tau(g^{-1}) = \eta' g^T \eta'^{-1} \tag{4.30}$$

with

$$\eta' = \mathrm{diag}\,(-1,-1,1,1,-1,-1,1,1) \tag{4.31}$$

the inner product preserved by $\mathrm{SO}(2,2)\times\mathrm{SO}(2,2)$. The group element $V(x)$ that we choose as the "triangular" coset representative is given by

$$V = e^{-UH_0}\cdot\left[\prod_{I=1,2,3}\left(e^{-\frac{1}{2}\log y^I H_I} e^{-x^I E_I}\right)\right]\cdot e^{-\zeta^\Lambda E_{q_\Lambda}-\tilde{\zeta}_\Lambda E_{p^\Lambda}}\cdot e^{-\sigma E_0}. \tag{4.32}$$

In the above expression, $H_0, H_I, E_I, E_{q_\Lambda}, E_{p^\Lambda}, E_0$ are generators of $\mathrm{SO}(4,4)$. We will use the fundamental representation basis in [61, 58]. The defining property of matrices $g$ in $\mathrm{SO}(4,4)$ is

$$g^{-1} = \eta g^T \eta \tag{4.33}$$

with $\eta$ the invariant metric given by

$$\eta = \begin{pmatrix} 0_4 & \mathbb{1}_4 \\ \mathbb{1}_4 & 0_4 \end{pmatrix}. \tag{4.34}$$

We will group the 28 generators of $\mathrm{SO}(4,4)$ as follows ($\Lambda = 0,1,2,3$)

$$H_\Lambda, \quad E_\Lambda, \quad F_\Lambda, \quad E_{q_\Lambda}, \quad F_{q_\Lambda}, \quad E_{p^\Lambda}, \quad F_{p^\Lambda} \tag{4.35}$$

and write them in terms of $8\times 8$ matrices $E_{ij}$ whose $(ij)$-entry is one and all the rest are zero [61]:

$$\begin{aligned} H_0 &= E_{33}+E_{44}-E_{77}-E_{88}, & H_1 &= E_{33}-E_{44}-E_{77}+E_{88}, \\ H_2 &= E_{11}+E_{22}-E_{55}-E_{66}, & H_3 &= E_{11}-E_{22}-E_{55}+E_{66}, \end{aligned} \tag{4.36}$$

$$E_0 = E_{47}-E_{38}, \quad E_1 = E_{87}-E_{34}, \quad E_2 = E_{25}-E_{16}, \quad E_3 = E_{65}-E_{12}, \tag{4.37}$$

$$F_0 = E_{74}-E_{83}, \quad F_1 = E_{78}-E_{43}, \quad F_2 = E_{52}-E_{61}, \quad F_3 = E_{56}-E_{21}, \tag{4.38}$$

$$E_{q_0} = E_{41} - E_{58}, \quad E_{q_1} = E_{57} - E_{31}, \quad E_{q_2} = E_{46} - E_{28}, \quad E_{q_3} = E_{42} - E_{68}, \quad (4.39)$$

$$F_{q_0} = E_{14} - E_{85}, \quad F_{q_1} = E_{75} - E_{13}, \quad F_{q_2} = E_{64} - E_{82}, \quad F_{q_3} = E_{24} - E_{86}, \quad (4.40)$$

$$E_{p^0} = E_{17} - E_{35}, \quad E_{p^1} = E_{18} - E_{45}, \quad E_{p^2} = E_{67} - E_{32}, \quad E_{p^3} = E_{27} - E_{36}, \quad (4.41)$$

$$F_{p^0} = E_{71} - E_{53}, \quad E_{p^1} = E_{81} - E_{54}, \quad F_{p^2} = E_{76} - E_{23}, \quad F_{p^3} = E_{72} - E_{63}. \quad (4.42)$$

Moreover, the subgroup $\mathrm{SO}(2,2) \times \mathrm{SO}(2,2) \approx \mathrm{SL}(2)^4$ is generated by

$$K_\Lambda = E_\Lambda - F_\Lambda, \quad K_{q_\Lambda} = E_{q_\Lambda} + F_{q_\Lambda}, \quad K_{p^\Lambda} = E_{p^\Lambda} + F_{p^\Lambda}. \tag{4.43}$$

Finally, we make the note that for the generators (4.36), we have that $\mathrm{Tr}(H_\Lambda H_\Lambda) = 4$, which due to the convention (2.38) in section 2.2 gives

$$\langle P_m, P_n \rangle = \mathrm{Tr}(P_m P_n) \tag{4.44}$$

for the matrices $P_m$ in the group $\mathrm{SO}(4,4)$ in the basis stated above. We note that the factor in front of the trace is now one instead of two in (2.51). The equations of motion derived from (4.27) are (2.39a),(2.39b) with the relation (4.44).

## 4.2 Riemann-Hilbert factorisation for $\mathrm{SO}(4,4)$

So far we have discussed the STU supergravity theory up to the point of reduction to three dimensions, whereby the (finite) Ehlers symmetry arises. The problem of finding solutions of this theory becomes tractable by the inverse scattering method when the theory is reduced to two dimensions, along an additional (spacelike) Killing direction. The equations of motion in two dimensions are formally identical to (2.55a),(2.55b). Specifically, due to (4.44) we get

$$\pm i f_E^{-1} \partial_\pm f_E = \frac{\rho}{4} \mathrm{Tr}(P_\pm P_\pm) \tag{4.45a}$$

$$D_\mu(\rho P^\mu) = 0, \tag{4.45b}$$

where the only change with respect to chapter 3 is the factor (1/4) on the right hand side of (4.45a) instead of (1/2) in (3.69). The solutions are of course restricted to the class of stationary, axisymmetric spacetimes. However, this is physically quite an interesting sector, since it includes black hole solutions that are attainable by the BM soliton algorithm.

In this section we will extend the BM technique to solitonic matrices $\mathcal{M}(w) \in \mathrm{SO}(4,4)$. We will proceed along the lines of section 3.2, extending it to the case of $\mathrm{G}_E = \mathrm{SO}(4,4)$, $\mathrm{K}_E = \mathrm{SO}(2,2) \times \mathrm{SO}(2,2)$ that is relevant to the STU model discussed above. We will present the formalism in the Ehlers coset only and will drop the subscript "E" from the matrix-valued functions to simplify our notation. For the case of *four-dimensional, asymptotically flat solutions*, we will take our seed solution to be flat space with

$$\mathcal{V}(t) = \mathbb{1} \quad \text{and} \quad f = 1 \tag{4.46}$$

where $\mathcal{V}$ is an $8 \times 8$ matrix $\in \mathrm{SO}(4,4)$ and $t$ denotes the BM spectral parameter (not to be confused with the time coordinate). One interesting modification that arises in this context is that we take the residue matrices to be of *rank two*. This is supported by the observation that for physically interesting solutions, the matrix $M(x)$ appears to "contain" the corresponding SL(2) matrix (of Einstein gravity) *twice*. In particular, we observe this structure of $M(x)$ (and $\mathcal{M}(w)$) in known solutions such as the Kerr solution in the context of STU gravity. We find that this is still true for charged rotating black hole solutions. The latter can be obtained from Harrison transformations of the Kerr coset metric $M(x)$ [58]. We will explicitly reconstruct this solution through charging transformations of the 2-soliton Kerr monodromy matrix $\mathcal{M}(w)$ that we factorise using the BM algorithm that follows.

We start again from the $\sharp$-invariant monodromy matrix

$$\mathcal{M}(w) = \mathcal{V}^\sharp\left(-\frac{1}{t}, x\right)\mathcal{V}(t,x) \tag{4.47}$$

and defining a matrix $A_+(t,x)$ as in section 3.2

$$\mathcal{V}(t,x) = V(x)A_+(t,x) \tag{4.48}$$

$$(\mathcal{V}(t,x))^\sharp = A_+^\sharp(-\frac{1}{t}, x)V^\sharp(x) \tag{4.49}$$

with $A_+, A_- \in \mathrm{SO}(4,4)$ and $A_-(t,x) = A_+\left(-\frac{1}{t}, x\right)$, we wish to factorize the monodromy matrix $\mathcal{M}$ as

$$\mathcal{M}(w) = A_-^\sharp(t,x)M(x)A_+(x). \tag{4.50}$$

As before, we consider the solitonic ansatz for $\mathcal{M}(w)$ (3.46), but due to the property (4.33), we get

$$\mathcal{M}(w) = \mathbb{1} + \sum_{k=1}^{N} \frac{A_k}{w - w_k}, \tag{4.51a}$$

$$\mathcal{M}^{-1}(w) = \eta\mathcal{M}^T\eta = \eta\left(\mathbb{1} + \sum_{k=1}^{N} \frac{A_k^T}{w - w_k}\right)\eta. \tag{4.51b}$$

Using the expression (3.52) we arrive at

$$\mathcal{M}(t,x) = \mathbb{1} + \sum_{k=1}^{N} \frac{\nu_k t_k A_k}{t-t_k} + \sum_{k=1}^{N} \frac{\nu_k A_k}{1+tt_k}\,. \tag{4.52}$$

We factorize the rank-two residue matrices as

$$A_k = \alpha_k a_k a_k^T \eta' - \beta_k (\eta b_k)(\eta b_k)^T \eta', \tag{4.53}$$

where $a_k, b_k$ are constant 8-dimensional vectors and $\alpha_k, \beta_k$ are constant parameters. As we will see shortly, the above choice of notation for the residues is explained if one keeps in mind the $\mathrm{SL}(2,\mathbb{R})$ case of the previous chapter. Moreover, it is easy to see that the matrices (4.53) are $\sharp$-invariant, as required for the $\sharp$-invariance of $\mathcal{M}(w)$. Following the same type of analysis as in section 3.2, we study the conditions for absence of single and double poles at $t = -\frac{1}{t_k}$ in the product $\mathcal{M}\mathcal{M}^{-1}$ or in this case the product $\mathcal{M}\eta\mathcal{M}^T$.

The conditions for no double poles read

$$A_k \eta A_k^T = 0 \quad \text{for all } k\,. \tag{4.54}$$

or in terms of the vectors $a_k, b_k$

$$a_k^T \eta a_k = 0, \tag{4.55a}$$
$$b_k^T \eta b_k = 0, \tag{4.55b}$$
$$a_k^T b_k = 0, \tag{4.55c}$$

for all $k$. Next, we write the condition for no single poles, that is

$$\left(\mathcal{M}(t,x) - \frac{\nu_k A_k}{1+tt_k}\right)\Bigg|_{t\to -\frac{1}{t_k}} \eta A_k^T = -A_k \eta \left(\mathcal{M}(t,x) - \frac{\nu_k A_k}{1+tt_k}\right)^T\Bigg|_{t\to -\frac{1}{t_k}}\,. \tag{4.56}$$

Defining matrices $\mathcal{A}_k$ as

$$\mathcal{A}_k = \left(\mathcal{M}(t,x) - \frac{\nu_k A_k}{1+tt_k}\right)\Bigg|_{t\to -\frac{1}{t_k}}\,, \tag{4.57}$$

we write the condition (4.56) as

$$\mathcal{A}_k \eta\eta' \alpha_k a_k a_k^T - \mathcal{A}_k \eta\eta' \beta_k (\eta b_k)(\eta b_k)^T = -\alpha_k a_k a_k^T \eta' \eta \mathcal{A}_k^T + \beta_k (\eta b_k)(\eta b_k)^T \eta' \eta \mathcal{A}_k^T\,. \tag{4.58}$$

For the above condition to hold, it is sufficient that there exist (space-time dependent) $\gamma_k$ such that

$$\mathcal{A}_k \eta\eta' a_k = \nu_k \beta_k \gamma_k (\eta b_k), \tag{4.59a}$$
$$(\eta b_k)^T \eta' \eta \mathcal{A}_k^T = \nu_k \alpha_k \gamma_k a_k^T\,. \tag{4.59b}$$

Next, we write a generalised ansatz for the matrix $A_+$ (cf. (3.57))

$$A_+(t) = \mathbb{1} - \sum_{k=1}^{N} \frac{tC_k}{1+tt_k}\,, \tag{4.60}$$

where the matrices $C_k$ are decomposed as

$$C_k = c_k a_k^T \eta' - (\eta d_k)(\eta b_k)^T \eta'\,. \tag{4.61}$$

Again, as in the SL$(n,\mathbb{R})$ case, we study the poles in the product $A_+(t)\eta\mathcal{M}^T(t,x)$ at $t=-1/t_k$, in order to find the relations of the vectors $a_k, b_k, c_k$, and $d_k$. The requirement that the product have no double poles results in the condition

$$C_k \eta A_k^T = 0\,, \tag{4.62}$$

which is satisfied when (4.55) hold. The absence of single poles at $t=-1/t_k$ in $A_+(t)\eta\mathcal{M}^T(t,x)$ leads to the condition

$$t_k^{-1} C_k \eta \mathcal{A}_k^T + \left( A_+ + \frac{tC_k}{1+tt_k} \right)\Bigg|_{t=-\frac{1}{t_k}} \eta \nu_k A_k^T = 0. \tag{4.63}$$

The above equation becomes

$$\begin{aligned} &t_k^{-1}\left(c_k\nu_k\beta_k\gamma_k(\eta b_k)^T - (\eta d_k)\nu_k\alpha_k\gamma_k a_k^T\right) + \nu_k\alpha_k\eta\eta' a_k a_k^T - \nu_k\beta_k\eta\eta'(\eta b_k)(\eta b_k)^T \\ &+\sum_{\substack{l=1\\ l\neq k}}^{N} \frac{1}{t_k-t_l}\left(c_l a_l^T\eta' - (\eta d_l)(\eta b_l)^T\eta'\right)\eta\nu_k\left(\eta'\alpha_k a_k a_k^T - \eta'\beta_k(\eta b_k)(\eta b_k)^T\right) = 0, \end{aligned} \tag{4.64}$$

where we used relations (4.59a) and (4.59b). We split the above condition into two sufficient conditions, namely

$$-t_k^{-1}(\eta d_k)\nu_k\alpha_k\gamma_k + \nu_k\alpha_k\eta\eta' a_k + \sum_{\substack{l=1\\ l\neq k}}^{N} \frac{\nu_k\alpha_k}{t_k-t_l}\left(c_l a_l^T\eta a_k - (\eta d_l)(\eta b_l)^T\eta a_k\right) = 0, \tag{4.65}$$

and

$$t_k^{-1}c_k\nu_k\beta_k\gamma_k - \nu_k\beta_k\eta\eta'(\eta b_k) - \sum_{\substack{l=1\\ l\neq k}}^{N} \frac{\nu_k\beta_k}{t_k-t_l}\left(c_l a_l^T\eta(\eta b_k) - (\eta d_l)(\eta b_l)^T\eta(\eta b_k)\right) = 0. \tag{4.66}$$

If we assume further orthogonality conditions for the vectors $a_k$, $b_k$, i.e.

$$a_l^T\eta a_k = 0, \tag{4.67a}$$

$$b_l^T\eta b_k = 0, \tag{4.67b}$$

for $l \neq k$, we reach the vector equations

$$\eta' a_k = \frac{\gamma_k}{t_k} d_k + \sum_{l \neq k}^{N} \frac{1}{t_k - t_l} d_l \left(a_k^T b_l\right), \tag{4.68}$$

$$\eta' b_k = \frac{\gamma_k}{t_k} c_k + \sum_{l \neq k}^{N} \frac{1}{t_l - t_k} c_l \left(a_l^T b_k\right). \tag{4.69}$$

which in matrix form read

$$\eta' a = d\, \Gamma^T, \tag{4.70a}$$

$$\eta' b = c\, \Gamma. \tag{4.70b}$$

The matrices $a, b, c$, and $d$ are $8 \times N$ matrices whose columns are the vectors $a_k, b_k, c_k$, $d_k$ respectively and $\Gamma$ is a $N \times N$ matrix with elements

$$\Gamma_{kl} = \begin{cases} \frac{\gamma_k}{t_k} & \text{for} \qquad k = l \\ \frac{a_k^T b_l}{t_k - t_l} & \text{for} \qquad k \neq l. \end{cases} \tag{4.71}$$

With the above relations, we may write the matrix $A_+$ as

$$A_+(t) = \mathbb{1} - \eta' b \Gamma^{-1} \frac{t}{\mathbb{1} + tT} a^T \eta' + \eta \eta' a \left(\Gamma^T\right)^{-1} \frac{t}{\mathbb{1} + tT} b^T \eta \eta', \tag{4.72}$$

where $T$ is the $N \times N$ diagonal matrix with entries $t_k$. Finally, the matrix $M(x)$ is given by the limit of the inverse of (4.72) as $t \to \infty$, i.e.

$$M(x) = A_+^{-1}(\infty) = \eta A_+^T(\infty) \eta, \tag{4.73}$$

with

$$A_+^T(\infty) = \mathbb{1} - \eta' a T^{-1} \left(\Gamma^{-1}\right)^T b^T \eta' + \eta' \eta b T^{-1} \Gamma^{-1} a^T \eta' \eta. \tag{4.74}$$

With the additional assumption $a_l^T b_k = -a_k^T b_l$ for $l \neq k$, i.e., that the $\Gamma$ matrix is symmetric, (4.74) becomes

$$A_+^T(\infty) = \mathbb{1} - \eta' a T^{-1} \Gamma^{-1} b^T \eta' + \eta' \eta b T^{-1} \Gamma^{-1} a^T \eta' \eta. \tag{4.75}$$

We find that for solutions such as the Kerr black hole and the four-charge rotating black holes in STU supergravity, the assumptions in this section hold true. It might be the case that this remains true for solutions of physical interest in general.

### Computation of the conformal factor

In the previous chapter, right after section 3.2, we outlined the algebraic computation of the conformal factor formula in the soliton sector (cf. (3.76)). The process is generally the same but some differences should be noted. In the present case, the matrix $A_+$ is of rank-two and is given by (4.60), (4.61). Moreover, the equation for the conformal factor (4.45a) has now a factor of (1/4) in front of $\mathrm{Tr}(P_\pm P_\pm)$ (instead of (1/2)). Therefore equation (3.72) becomes

$$\partial_\pm \ln f_E = -\frac{1}{4}(\partial_\pm \ln\rho)\mathrm{Tr}\left(A_+^{-1}(\pm i)\dot{A}_+(\pm i)\right)^2. \tag{4.76}$$

Substituting the final expression (4.72) for $A_+$, the product $A_+^{-1}(t)\frac{\partial}{\partial t}A_+(t)$ in the above equation is given by

$$A_+^{-1}(t)\frac{\partial}{\partial t}A_+(t) = -\eta' b\frac{\mathbb{1}}{\mathbb{1}+tT}\Gamma^{-1}\frac{\mathbb{1}}{\mathbb{1}+tT}a^T\eta' + \eta\eta' a\frac{\mathbb{1}}{\mathbb{1}+tT}\Gamma^{-1}\frac{\mathbb{1}}{\mathbb{1}+tT}b^T\eta\eta' \tag{4.77}$$

and the trace on the right hand side of (4.76) becomes

$$\mathrm{Tr}(A_+^{-1}(\pm i)\dot{A}_+(\pm i))^2 = 2\sum_{k,l,m,n}\frac{\Gamma^{-1}_{kl}\Gamma^{-1}_{mn}}{(1\pm it_k)(1\pm it_l)(1\pm it_m)(1\pm it_n)}\mathrm{Tr}(b_k a_l^T b_m a_n^T). \tag{4.78}$$

where the factor of 2 on the right hand side is due to the increased rank of matrix $A_+$. We find that the conformal factor is given by

$$f^2 = k_{BM}\cdot\prod_{k=1}^{N}(t_k\nu_k)\cdot\det\Gamma, \tag{4.79}$$

where $k_{BM}$ is an integration constant.

### 4.2.1 Construction of four-charge rotating black hole

In this section we will apply the algorithm presented above to construct a four-charge rotating black hole, with flat space as the seed solution.

We start with a 2-soliton ansatz for the monodromy matrix

$$\mathcal{M}(w) = \mathbb{1} + \frac{A_1}{w-c} + \frac{A_2}{w+c}, \tag{4.80}$$

where the residue matrices are given by

$$A_1 = \alpha_1 a_1 a_1^T\eta' - \beta_1(\eta b_1)(\eta b_1)^T\eta', \tag{4.81a}$$
$$A_2 = \alpha_2 a_2 a_2^T\eta' - \beta_2(\eta b_2)(\eta b_2)^T\eta', \tag{4.81b}$$

and where $a_1, a_2$ and $b_1, b_2$ are 8-dimensional vectors. In order to find the vectors $a_1, a_2$ and $b_1, b_2$ for the four-charge black hole, we look at the simpler case of the Kerr-black hole in the SO(4, 4) context. From the structure of the SO(4, 4) matrix $M(x)$ and embedding of the Ehlers's $\mathrm{SL}(2,\mathbb{R})$ in it, we make the ansatz for the vectors $a_1, a_2$

$$a_1 = (0,0,-\zeta,0,0,0,0,1)^T, \tag{4.82a}$$
$$a_2 = (0,0,1,0,0,0,0,-\zeta)^T. \tag{4.82b}$$

For the $b$-vectors, similarly to the previous chapter, we construct the $\xi = a^T\eta' a$ matrix, with $a$ the $8\times 2$ matrix whose columns are the vectors $a_1, a_2$. The corresponding $b$ matrix is given by

$$b = (\sqrt{\det\xi})\eta' a\xi^{-1}\epsilon \qquad \text{with} \qquad \epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.83}$$

We note that the factor $\sqrt{\det\xi}$ above is included so that the $b$-vectors acquire a simpler form. We get

$$\begin{aligned} b_1 &= (0,0,1,0,0,0,0,\zeta)^T & (4.84a)\\ b_2 &= (0,0,-\zeta,0,0,0,0,-1)^T. & (4.84b)\end{aligned}$$

The parameters $\alpha_k, \beta_k$ must be chosen such that $\mathcal{M}$ has unit determinant. We find

$$\alpha_1 = +2c\frac{1+\zeta^2}{(1-\zeta^2)^2}, \qquad \alpha_2 = -2c\frac{1+\zeta^2}{(1-\zeta^2)^2}, \tag{4.85}$$

$$\beta_1 = -2c\frac{1+\zeta^2}{(1-\zeta^2)^2}, \qquad \beta_2 = +2c\frac{1+\zeta^2}{(1-\zeta^2)^2}, \tag{4.86}$$

Moreover, we can easily verify that the conditions (4.55) as well as the additional assumptions (4.67) and $a_l^T b_k = -a_k^T b_l$ for $l \neq k$ hold. We have that

$$\begin{aligned} &a_1^T\eta a_1 = 0, && a_2^T\eta a_2 = 0, && a_1^T\eta a_2 = 0, && (4.87a)\\ &b_1^T\eta b_1 = 0, && b_2^T\eta b_2 = 0, && b_1^T\eta b_2 = 0, && (4.87b)\\ &a_1^T b_1 = 0, && a_2^T b_2 = 0, && a_1^T b_2 = -a_2^T b_1 = -1+\zeta^2. && (4.87c)\end{aligned}$$

Identifying the parameters $\zeta, c$ as follows

$$\zeta = \frac{c-m}{a}, \qquad c = \sqrt{m^2-a^2}. \tag{4.88}$$

we find that the monodromy matrix reads

$$\mathcal{M}(w) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1+\frac{2m(m-w)}{w^2-c^2} & 0 & 0 & 0 & 0 & \frac{2am}{w^2-c^2}\\ 0 & 0 & 0 & 1+\frac{2m(m-w)}{w^2-c^2} & 0 & 0 & -\frac{2am}{w^2-c^2} & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -\frac{2am}{w^2-c^2} & 0 & 0 & 1+\frac{2m(m+w)}{w^2-c^2} & 0\\ 0 & 0 & \frac{2am}{w^2-c^2} & 0 & 0 & 0 & 0 & 1+\frac{2m(m+w)}{w^2-c^2} \end{pmatrix} \tag{4.89}$$

The factorisation of the above matrix corresponds to the Kerr solution in the context of the STU model. Given the Kerr monodromy matrix, we can reach a monodromy matrix for a charged black hole solution through appropriate group transformations as follows [28, 58, 51]

$$\mathcal{M}_{\text{4-charge}}(w) = g^\sharp \mathcal{M}(w) g. \tag{4.90}$$

In particular, for the four-charge black hole with three-magnetic charges and one-electric charge, the charging group element is

$$g = \exp[-\delta_0(E_{q_0} + F_{q_0})] \cdot \exp[\delta_1(E_{p^1} + F_{p^1})] \cdot \exp[\delta_2(E_{p^2} + F_{p^2})] \cdot \exp[\delta_3(E_{p^3} + F_{p^3})], \tag{4.91}$$

which is an element in $\mathrm{K_E}$, satisfying $g^\sharp g = 1$ (cf.(4.43)). Applying this group transformation on (4.80) with vectors given by (4.82), we find that the new vectors read

$$\begin{aligned}
a_1 &= (-c_0 s_1, -\zeta c_3 s_2, -\zeta c_2 c_3, -s_0 s_1, -c_1 s_0, -\zeta c_2 s_3, \zeta s_2 s_3, c_0 c_1)^T, && (4.92\text{a})\\
a_2 &= (\zeta c_0 s_1, c_3 s_2, c_2 c_3, \zeta s_0 s_1, \zeta c_1 s_0, c_2 s_3, -s_2 s_3, -\zeta c_0 c_1)^T, && (4.92\text{b})\\
b_1 &= (\zeta c_0 s_1, -c_3 s_2, c_2 c_3, -\zeta s_0 s_1, \zeta c_1 s_0, -c_2 s_3, -s_2 s_3, \zeta c_0 c_1)^T, && (4.92\text{c})\\
b_2 &= (-c_0 s_1, \zeta c_3 s_2, -\zeta c_2 c_3, s_0 s_1, -c_1 s_0, \zeta c_2 s_3, \zeta s_2 s_3, -c_0 c_1)^T, && (4.92\text{d})
\end{aligned}$$

where we use the notation $c_i = \cosh\delta_i$ and $s_i = \sinh\delta_i$. The conditions (4.87a), (4.87c) remain true by virtue of group properties. Next, we solve (4.68) for $\gamma_k$ and find

$$\begin{aligned}
\gamma_1 &= \tfrac{2\zeta(1-\zeta^2)t_2(1+t_1^2)}{(1+\zeta^2)(t_1-t_2)(1+t_1t_2)}, && (4.93\text{a})\\
\gamma_2 &= \tfrac{2\zeta(1-\zeta^2)t_1(1+t_2^2)}{(1+\zeta^2)(t_1-t_2)(1+t_1t_2)}. && (4.93\text{b})
\end{aligned}$$

With the above expressions we construct the $\Gamma$ matrix from the definition (4.71) and subsequently the $c$ and $d$ matrices (hence the $c_k$ and $d_k$ vectors). Thus we have achieved the factorisation (4.50) for the charged monodromy matrix and we obtain $M(x)$ through (4.73),(4.75). From the matrix $M$ we read off the scalar fields and once we integrate all the relevant duality relations to find the scalars that enter the final metric, we are able to reconstruct the corresponding part of the four-charge black hole metric. For the full solution, we need the conformal factor which we get using (4.79):

$$f^2 = -4k_{BM}t_1^2t_2^2(1-\zeta^2)^2 \frac{(1+t_1t_2)^2(1-\zeta^2)^2 - 4(t_1-t_2)^2\zeta^2}{(1+t_1^2)(1+t_2^2)(t_1-t_2)^2(1+t_1t_2)^2(1+\zeta^2)^2\rho^2}. \tag{4.94}$$

### Getting to the four-charge black hole

After successful factorization of (4.90), we are left with the matrix $M(x)$, from which we read off the scalar fields $\varphi^i$. In principle, together with the conformal factor (4.94), we have all the information needed to construct the line element of the four-charge black hole geometry. However, there are a few more steps into the "decoding" of all this information and its final, "recognizable" form.

Starting from the $8 \times 8$ matrix $M^{(\text{4-charge})}$ which is a function of two variables $(t_1, t_2)$ (i.e. the pole trajectories), we first change to the more convenient prolate coordinates $(u, v)$ using relations (3.95). Next, according to the parameterization (4.32), we read off the profiles for the various scalar fields and perform the integrations of equations (4.23),(4.24). From these integrations we obtain the one-forms $\omega_3, A_3^\Lambda$ and we can write the metric in the form (4.21) (or the five-dimensional uplift given by (4.10)). In order to recognize the metric as that of a (charged), rotating black hole, we identify the parameters $\zeta, c$ as in (4.88) (in these expressions $a$ is the bare rotation parameter and $m$ is the bare mass parameter) and change to Boyer-Lindquist coordinates (3.103) for the presentation of the final expressions. We first write these expressions using the following definitions [1]

$$
\begin{aligned}
\Delta &= \frac{r^2 + a^2x^2 - 2mr}{r^2 + a^2x^2}, && (4.95)\\
\sigma_{\text{Kerr}} &= -\frac{2max}{r^2 + a^2x^2}, && (4.96)\\
W^2 &= h_0h_1h_2h_3 + \sigma_{\text{Kerr}}^2\big(2c_{0123}s_{0123} - (s_{012}^2 + s_{013}^2 + s_{023}^2 + s_{123}^2 + 4s_{0123}^2)\Delta \\
&\quad + 2s_{0123}^2\Delta^2\big) + s_{0123}^2\sigma_{\text{Kerr}}^4. && (4.97)
\end{aligned}
$$

where we have set $x \equiv \cos\theta$ and

$$
\begin{aligned}
h_i &= (c_i^2 - s_i^2\Delta), && (4.98a)\\
c_{i_1\ldots i_n} &= \cosh\delta_{i_1}\ldots\cosh\delta_{i_n}, && (4.98b)\\
s_{i_1\ldots i_n} &= \sinh\delta_{i_1}\ldots\sinh\delta_{i_n}, && (4.98c)
\end{aligned}
$$

(the $h_i$ here are not to be confused with the fields $h^I$ in section 4.1). For the fields $x_I, y_I$ we find

$$
\begin{aligned}
x_1 &= \frac{(c_{01}s_{23} - s_{01}c_{23})\sigma_{\text{Kerr}}}{h_2h_3 + s_{23}^2\sigma_{\text{Kerr}}^2}, && (4.99a)\\
x_2 &= \frac{(c_{02}s_{13} - s_{02}c_{13})\sigma_{\text{Kerr}}}{h_1h_3 + s_{13}^2\sigma_{\text{Kerr}}^2}, && (4.99b)\\
x_3 &= \frac{(c_{03}s_{12} - s_{03}c_{12})\sigma_{\text{Kerr}}}{h_1h_2 + s_{12}^2\sigma_{\text{Kerr}}^2}, && (4.99c)\\
y_1 &= \frac{W}{h_2h_3 + s_{23}^2\sigma_{\text{Kerr}}^2}, && (4.99d)\\
y_2 &= \frac{W}{h_1h_3 + s_{13}^2\sigma_{\text{Kerr}}^2}, && (4.99e)\\
y_3 &= \frac{W}{h_1h_2 + s_{12}^2\sigma_{\text{Kerr}}^2} && (4.99f)
\end{aligned}
$$

[1] We note that, our result for $\sigma_{\text{Kerr}}$ matches the corresponding result in [58] when multiplied by a factor of 2. This is because in [58] there is a factor of 1/2 multiplying the $\sigma$ field in the parameterization of $V(x)$.

and for the dilaton $U$ and the field $\sigma$ we get

$$U = \frac{1}{2} \ln \frac{\Delta}{W}, \tag{4.100}$$

$$\sigma = \frac{\sigma_{\rm Kerr}}{2W^2} \Big\{ c_{0123} \left[ 2 + (1-\Delta) \left( \sum_{i=0}^{3} s_i^2 \right) \right] + s_{0123} \Big[ \left( 2 + \sum_{i=0}^{3} s_i^2 \right) \\ (\Delta^2 - \Delta + \sigma_{\rm Kerr}^2) - 2\Delta \Big] \Big\}. \tag{4.101}$$

The rest of the scalars $\zeta^\Lambda, \tilde{\zeta}_\Lambda$ are

$$\tilde{\zeta}_0 = \frac{\sigma_{\rm Kerr}}{W^2} \left[ h_0 (s_0 c_{123} - c_0 s_{123} \Delta) + s_0 c_0 s_{0123} \sigma_{\rm Kerr}^2 \right], \tag{4.102a}$$

$$\zeta^1 = \frac{\sigma_{\rm Kerr}}{W^2} \left[ h_1 (s_1 c_{023} - c_1 s_{023} \Delta) + s_1 c_1 s_{0123} \sigma_{\rm Kerr}^2 \right], \tag{4.102b}$$

$$\zeta^2 = \frac{\sigma_{\rm Kerr}}{W^2} \left[ h_2 (s_2 c_{013} - c_2 s_{013} \Delta) + s_2 c_2 s_{0123} \sigma_{\rm Kerr}^2 \right], \tag{4.102c}$$

$$\zeta^3 = \frac{\sigma_{\rm Kerr}}{W^2} \left[ h_3 (s_3 c_{012} - c_3 s_{012} \Delta) + s_3 c_3 s_{0123} \sigma_{\rm Kerr}^2 \right], \tag{4.102d}$$

and

$$\zeta^0 = + \left\{ \tfrac{c_0}{s_0} - \tfrac{1}{s_0 W^2} (c_0 h_1 h_2 h_3 + (s_0 c_{123} - c_0 s_{123} \Delta) s_{123} \sigma_{\rm Kerr}^2) \right\}, \tag{4.103a}$$

$$\tilde{\zeta}_1 = - \left\{ \tfrac{c_1}{s_1} - \tfrac{1}{s_1 W^2} (c_1 h_0 h_2 h_3 + (s_1 c_{023} - c_1 s_{023} \Delta) s_{023} \sigma_{\rm Kerr}^2) \right\}, \tag{4.103b}$$

$$\tilde{\zeta}_2 = - \left\{ \tfrac{c_2}{s_2} - \tfrac{1}{s_2 W^2} (c_2 h_0 h_1 h_3 + (s_2 c_{013} - c_2 s_{013} \Delta) s_{013} \sigma_{\rm Kerr}^2) \right\}, \tag{4.103c}$$

$$\tilde{\zeta}_3 = - \left\{ \tfrac{c_3}{s_3} - \tfrac{1}{s_3 W^2} (c_3 h_0 h_1 h_2 + (s_3 c_{012} - c_3 s_{012} \Delta) s_{012} \sigma_{\rm Kerr}^2) \right\}. \tag{4.103d}$$

Finally, the one-forms $\omega_3, A_3^\Lambda$ are given by

$$\omega_3 = 2am(1-x^2) \frac{(c_{0123} r - (r-2m) s_{0123})}{r^2 - 2mr + a^2 x^2} d\phi, \tag{4.104}$$

and

$$A_3^0 = -2am(1-x^2) \frac{(s_0 c_{123} r - (r-2m) c_0 s_{123})}{r^2 - 2mr + a^2 x^2} d\phi, \tag{4.105}$$

$$A_3^1 = 2m s_1 c_1 x \frac{r^2 + a^2 - 2mr}{r^2 - 2mr + a^2 x^2} d\phi, \tag{4.106}$$

$$A_3^2 = 2m s_2 c_2 x \frac{r^2 + a^2 - 2mr}{r^2 - 2mr + a^2 x^2} d\phi, \tag{4.107}$$

$$A_3^3 = 2m s_3 c_3 x \frac{r^2 + a^2 - 2mr}{r^2 - 2mr + a^2 x^2} d\phi. \tag{4.108}$$

Together with the above one-forms, the list of fields involved in the final solution, after substituting $\sigma_{\text{Kerr}}, \Delta$, read

$$x_1 = 2max\frac{s_{01}c_{23} - c_{01}s_{23}}{r_2r_3 + a^2x^2}, \tag{4.109a}$$

$$x_2 = 2max\frac{s_{02}c_{13} - c_{02}s_{13}}{r_1r_3 + a^2x^2}, \tag{4.109b}$$

$$x_3 = 2max\frac{s_{03}c_{12} - c_{03}s_{12}}{r_1r_2 + a^2x^2}, \tag{4.109c}$$

where $r_i = r + 2ms_i^2$, and

$$y_1 = \frac{\tilde{W}}{r_2r_3 + a^2x^2}, \tag{4.110a}$$

$$y_2 = \frac{\tilde{W}}{r_1r_3 + a^2x^2}, \tag{4.110b}$$

$$y_3 = \frac{\tilde{W}}{r_1r_2 + a^2x^2}, \tag{4.110c}$$

with $\tilde{W}^2 := (r^2 + a^2x^2)^2W^2$ given by

$$\begin{aligned}\tilde{W}^2 &= r_0r_1r_2r_3 + a^4x^4 + a^2x^2[2r^2 + 2mr(s_0^2 + s_1^2 + s_2^2 + s_3^2)\\ &\quad + 8m^2c_{0123}s_{0123} - 4m^2(s_{012}^2 + s_{123}^2 + s_{023}^2s_{013}^2 + 2s_{0123}^2)].\end{aligned} \tag{4.111}$$

The scalars appearing in (4.22) are

$$\zeta^0 = \frac{2mc_0s_0(r_1r_2r_3 + ra^2x^2) + 4a^2m^2x^2e_0}{\tilde{W}^2}, \tag{4.112a}$$

$$\zeta^1 = -2max\frac{(s_1c_{023} - c_1s_{023})(rr_1 + a^2x^2) + 2mc_1s_{023}r_1}{\tilde{W}^2}, \tag{4.112b}$$

$$\zeta^2 = -2max\frac{(s_2c_{013} - c_2s_{013})(rr_2 + a^2x^2) + 2mc_2s_{013}r_2}{\tilde{W}^2}, \tag{4.112c}$$

$$\zeta^3 = -2max\frac{(s_3c_{012} - c_3s_{012})(rr_3 + a^2x^2) + 2mc_3s_{012}r_3}{\tilde{W}^2}, \tag{4.112d}$$

where

$$e_0 = (c_0^2 + s_0^2)c_{123}s_{123} - c_0s_0(s_{12}^2 + s_{23}^2 + s_{13}^2 + 2s_{123}^2). \tag{4.113}$$

Using these expressions, we can write the four-dimensional metric (4.21) and the various matter fields (4.22) that describe the four-charge Cvetic-Youm solution. Moreover, for the conformal factor given by (4.94), we choose the constant $k_{\text{BM}}$ to be

$$k_{\text{BM}} = -4c^2\frac{(1+\zeta^2)^2}{(1-\zeta^2)^4} = -\frac{m^2a^4}{c^2(m-c)^2}, \tag{4.114}$$

in order to ensure asymptotic flatness.

At this point there is one important note to make. As we have already seen, the monodromy of the four-charge solution is obtained by a group transformation of the form

$$\begin{aligned}\mathcal{M}^{(\text{4-charge})}(w) &= g^{\sharp}_{(\text{finite})}\, g^{\sharp}_{(\text{Geroch})}(w)\, \mathcal{M}^{(\text{flat})}\, g_{(\text{Geroch})}(w)\, g_{(\text{finite})} \\ &= g^{\sharp}_{(\text{finite})} \mathcal{M}^{(\text{Kerr})}(w) g_{(\text{finite})}, \end{aligned} \tag{4.115}$$

where $g^{\sharp}_{(\text{Geroch})}(w)\, \mathcal{M}^{(\text{flat})}\, g_{(\text{Geroch})}(w) = \mathcal{M}^{(\text{Kerr})}(w)$, i.e. the Geroch group transformation is the one supplying the pole structure that turns flat space into the Kerr black hole. The charges are added to this solution by the constant element in the finite subgroup $\mathrm{K}_E$. In the present case, as well as in all example solutions that we work out with the BM method, the explicit Geroch group transformation is not used. Instead, we start from a general meromorphic ansatz for $\mathcal{M}(w)$ that is an element in the affine extension of $\mathrm{G}_E$. However, if one wishes to view e.g. $\mathcal{M}^{(\text{Kerr})}(w)$ as a solution generated by a Geroch group transformation, the particular group element can be straightforwardly deduced[2] . As for the constant group transformation that we apply on $\mathcal{M}^{(\text{Kerr})}(w)$, it has the effect of rotating the vectors $a_k, b_k$ but does not affect the $\Gamma$ matrix (4.71). Indeed $\gamma_{1,2}$ in (4.93a),(4.93b) are the same as in the Kerr solution and the inner products of $a_k, b_k$ remain invariant under $\mathrm{K}_E$-transformations by virtue of the property $k^{\sharp}k = 1,\ k \in \mathrm{K}_E$. As a result, the conformal factor (cf. (4.79)) remains unchanged and so does the three-dimensional base metric in (4.21):

$$ds_3^2 = \frac{r^2 - 2mr + a^2x^2}{r^2 - 2mr + a^2} dr^2 + (r^2 - 2mr + a^2x^2)\frac{dx^2}{1 - x^2} + (1 - x^2)(r^2 - 2mr + a^2)d\phi^2. \tag{4.117}$$

This observation is consistent with the finite-dimensional solution generating transformations used by many authors (for example this is also noted in [63], where finite-group transformations are used to generate stationary axisymmetric solutions in five dimensional gravity).

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[2]Indeed, to generate the Kerr monodromy matrix (4.89), one acts on the flat space monodromy as $g^{\sharp}_{(\text{Geroch})}(w)\, \mathcal{M}^{(\text{flat})}\, g_{(\text{Geroch})}(w) = \mathcal{M}^{(\text{Kerr})}(w)$ with the Geroch group element $g(w)$ given by ($\mathcal{M}^{(\text{flat})} = \mathbb{1}$, $c^2 = m^2 - a^2$):

$$g(w) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{(w-m)^2+a^2}{w^2-c^2}} & 0 & 0 & 0 & 0 & \frac{-2am}{\sqrt{w^2-c^2}\sqrt{(w-m)^2+a^2}} \\ 0 & 0 & 0 & \sqrt{\frac{(w-m)^2+a^2}{w^2-c^2}} & 0 & 0 & \frac{2am}{\sqrt{w^2-c^2}\sqrt{(w-m)^2+a^2}} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{w^2-c^2}{(w-m)^2+a^2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{w^2-c^2}{(w-m)^2+a^2}} \end{pmatrix} \tag{4.116}$$

with $w$ in a suitable domain, where $\mathcal{M}^{(\text{Kerr})}(w)$ is holomorphic [13]. In this domain, the above $g$ is also holomorphic and satisfies $\det g_{(\text{Geroch})}(w) = 1$, $g_{(\text{Geroch})}(w \to \infty) = 1$.

## 4.3 Generalisation of Breitenlohner-Maison technique: residues of rank $r$

In this section, we will present a generalised version of the soliton BM algorithm that accounts for residue matrices of rank $r$. We assume a set-up where a symmetric space $\mathrm{G}_E/\mathrm{K}_E$ is relevant and that the group elements acquire a matrix representation. The anti-involution $\sharp$ is defined such that the elements $k$ in $\mathrm{K}_E$ satisfy $k^\sharp k = \mathbb{1}$.

Once again, we start with an $N$-soliton ansatz for $\mathcal{M}(w)$

$$\mathcal{M}(w) = \mathbb{1} + \sum_{k=1}^{N} \frac{A_k}{w - w_k}, \tag{4.118}$$

and

$$\mathcal{M}^{-1}(w) = \mathbb{1} - \sum_{k=1}^{N} \frac{B_k}{w - w_k}\,, \tag{4.119}$$

with $A_k, B_k$ the $x$-independent residue matrices. Let us also repeat the $t$-dependent expansions for $\mathcal{M}$

$$\mathcal{M}(t,x) = \mathbb{1} + \sum_{k=1}^{N} \frac{\nu_k t_k A_k}{t - t_k} + \sum_{k=1}^{N} \frac{\nu_k A_k}{1 + t t_k}, \tag{4.120}$$

and

$$\mathcal{M}^{-1}(t,x) = \mathbb{1} - \sum_{k=1}^{N} \frac{\nu_k t_k B_k}{t - t_k} - \sum_{k=1}^{N} \frac{\nu_k B_k}{1 + t t_k}\,. \tag{4.121}$$

We wish to factorise $\mathcal{M}$ with the above expansion, as in (4.50). We take $A_k$, $B_k$ to be $n \times n$ diagonalizable matrices of rank $r$, $(r \leq n)$, which are also $\sharp$-invariant, i.e. $A_k = A_k^\sharp$ and $B_k = B_k^\sharp$. Therefore, there exists a matrix $U_k$ with $U_k^{-1} = U_k^\sharp$ and a diagonal matrix $\Lambda_k$ such that

$$A_k = U_k \Lambda_k U_k^\sharp\,. \tag{4.122}$$

It follows that we can decompose the matrix $A_k$ (similarly $B_k$) as a sum of rank-one matrices as follows:

$$A_k = \sum_{\alpha=1}^{r} \lambda_k^\alpha u_k^\alpha v_k^{\alpha\,T}\,, \tag{4.123}$$

where $\lambda_k^\alpha$ are the entries of the diagonal matrix $\Lambda_k$ that are different than zero. By $u_k^\alpha$ and $v_k^{\alpha\,T}$ we denote the ($n$-dimensional) column vectors of matrix $U_k$ and corresponding row vectors of matrix $U_k^\sharp$ respectively, that correspond to the $\lambda_k^\alpha$. The decomposition (4.123) is not written in a manifestly"$\sharp$-invariant" form. However, it is possible to do so, as soon as we have the explicit action of the $\sharp$ operation on $g \in \mathrm{G}_E$. To illustrate this, let us take the already familiar example of the coset model $\mathrm{G}_E/\mathrm{K}_E = \mathrm{SO}(4,4)/(\mathrm{SO}(2,2)\times\mathrm{SO}(2,2))$. With the $\sharp$ acting on $g \in \mathrm{SO}(4,4)$ as given in (4.30), we write the residue matrices $A_k$ (similarly for $B_k$) as

$$\begin{aligned} A_k = U_k \Lambda_k U_k^{\sharp} = U_k \eta' \Lambda_k \eta' \eta' U^T \eta' = U_k \Lambda'_k U^T \eta' \\ = \sum_{\alpha=1}^{r} \lambda_k'^{\alpha} u_k^{\alpha} u_k^{\alpha\,\sharp}\,, \end{aligned} \tag{4.124}$$

where we use that $(\Lambda_k)^{\sharp} = \Lambda_k$ and the definition $\Lambda'_k = \eta' \Lambda_k$. Moreover, we define the $\sharp$ operation on column vectors as $u_k^{\sharp} = u_k^T \eta'$ and on row vectors as $u_k^{T\,\sharp} = \eta' u_k$. (With this definition , for any vector $v$ and a matrix $S$ given by $S = vv^{\sharp} \in \mathrm{G}_E$, we have that $S^{\sharp} = S$). Returning now to the general case and assuming we can write $A_k$ (similarly for $B_k$) as

$$A_k = \sum_{\alpha=1}^{r} \lambda_k'^{\alpha} u_k^{\alpha} u_k^{\alpha\,\sharp}\,, \tag{4.125}$$

we use the freedom to redefine the vectors and pull out suitable constant factors, so that we may write[3]

$$A_k = \alpha_k \sum_{\alpha=1}^{r} p_k^{\alpha} p_k^{\alpha\,\sharp}, \qquad B_k = \beta_k \sum_{\alpha=1}^{r} q_k^{\alpha} q_k^{\alpha\,\sharp}, \tag{4.126}$$

where we denote the redefined $n$-dimensional vectors as $p_k^{\alpha}, q_k^{\alpha}$ and $\alpha_k, \beta_k$ are constant parameters, not to be confused with the greek upper indices. (We note that here, we have one constant factor multiplying the rank-one decomposition of each residue matrix. We could also have constants enumerated by the rank index, multiplying each term of the sum - we avoid this here to keep the notation simpler). We use greek indices to enumerate the vectors with respect to the rank of the residue matrix, while the lower indices $k, l, ...$ are the soliton indices and take values in $\{1, 2, ..., N\}$. The algorithm that solves the factorisation problem, proceeds in the exact same way as before. We study the pole structure of the product $\mathcal{M}(t,x)\mathcal{M}^{-1}(t,x)$ at $t = -\frac{1}{t_k}$ and deduce the required conditions on the vectors $p_k^{\alpha}, q_k^{\alpha}$. We start with the condition for no double poles in the product which is

$$p_k^{\alpha\,\sharp} q_k^{\beta} = 0, \quad \text{for all } k \text{ and } \alpha = 1, 2, ..., r\,, \quad \beta = 1, 2, ..., r\,, \tag{4.127}$$

while the absence of single poles in $\mathcal{M}(t,x)\mathcal{M}^{-1}(t,x)$ at $t = -\frac{1}{t_k}$ is ensured by

$$\mathcal{A}_k B_k = A_k \mathcal{A}^k, \tag{4.128}$$

[3] In section 4.2, we have used different notation for the case of $\mathrm{G}/\mathrm{K} = \mathrm{SO}(4,4)/(\mathrm{SO}(2,2)\times\mathrm{SO}(2,2))$ coset. However, it is possible to translate everything in the general notation of the present section if we make the identifications $p_k^1 = a_k, p_k^2 = -\eta b_k, q_k^1 = \eta' b_k, q_k^2 = \eta\eta' a_k, \alpha_k^1 = -\beta_k^2 = \alpha_k, \alpha_k^2 = -\beta_k^1 = -\beta_k, r_k^1 = c_k, r_k^2 = \eta d_k, s_k^1 = \eta' d_k, s_k^2 = -\eta\eta' c_k$ (with $\alpha_k, \beta_k$ the constant parameters in 4.2 and using the $\sharp$ operation on vectors as defined above).

where $\mathcal{A}_k$,$\mathcal{A}^k$ are defined as follows

$$\mathcal{A}_k = \left(\mathcal{M}(t,x) - \frac{\nu_k A_k}{1+tt_k}\right)\Bigg|_{t\to -\frac{1}{t_k}}, \qquad \mathcal{A}^k = \left(\mathcal{M}^{-1}(t,x) + \frac{\nu_k B_k}{1+tt_k}\right)\Bigg|_{t\to -\frac{1}{t_k}}. \tag{4.129}$$

The condition (4.128) is fulfilled if there exist $\gamma_k^\alpha$ such that

$$\mathcal{A}_k q_k^\alpha = \nu_k \alpha_k \gamma_k^\alpha p_k^\alpha , \qquad p_k^{\alpha\,\sharp}\mathcal{A}^k = \nu_k \beta_k \gamma_k^\alpha q_k^{\alpha\,\sharp}, \tag{4.130}$$

for all $k = 1, 2, ..., N$ and $\alpha = 1, 2, ..., r$. Next, we turn to the matrix $A_+$ in (4.50) and choose the ansatz

$$A_+ = \mathbb{1} - \sum_{k=1}^{N} \frac{tC_k}{1+tt_k}, \tag{4.131}$$

and

$$A_+^{-1} = \mathbb{1} + \sum_{k=1}^{N} \frac{tD_k}{1+tt_k}, \tag{4.132}$$

where $C_k$ and $D_k$ are rank-$r$ matrices which also admit a decomposition of the form

$$C_k = \sum_{\alpha=1}^{r} r_k^\alpha p_k^{\alpha\,\sharp}, \qquad D_k = \sum_{\alpha=1}^{r} q_k^\alpha s_k^{\alpha\,\sharp}, \tag{4.133}$$

involving the vectors $r_k^\alpha, s_k^\alpha$ that correspond to the $c_k, d_k$ vectors that we have seen before. We find the relation that determines the vectors $r_k^\alpha$ from the pole structure of the product $A_+(t)\mathcal{M}^{-1}(t,x)$ at $t = -\frac{1}{t_k}$. The condition for no double poles gives

$$C_k B_k = 0, \tag{4.134}$$

and is fulfilled when (4.127) holds. Next, we write the condition for no single poles in the same product, that is

$$t_k^{-2} C_k \mathcal{A}^k = \left(A_+ + \frac{tC_k}{1+tt_k}\right)\Bigg|_{t\to -\frac{1}{t_k}} B_k \nu_k t_k^{-1}, \tag{4.135}$$

which can be fulfilled when

$$q_k^\alpha = t_k^{-1} r_k^\alpha \gamma_k^\alpha + \sum_{l\neq k}^{N}\sum_{\beta=1}^{r} \frac{1}{t_l - t_k} r_l^\beta p_l^{\beta\,\sharp} q_k^\alpha. \tag{4.136}$$

The above $rN$ vector equations can be written in the simpler form [4]

$$q_B = \sum_{A=1}^{rN} r_A \Gamma_{AB}\,, \tag{4.137}$$

[4] These vector equations can be represented by the matrix equation $q = r\,\Gamma$, where $q$ is the $n \times rN$ matrix whose columns are the vectors $q_1^1, q_2^1, ..., q_N^1, q_1^2, q_2^2, ..., q_N^2, ..., q_1^r, q_2^r, ..., q_N^r$ and the matrix $r$ is defined similarly (with columns the $r_k^\alpha$ vectors).

where we have merged each pair of indices $(k,\alpha)$ in a new index denoted by $A, B, ...$ that takes values in $\{1,2,...,rN\}$ .Each capital index is mapped to a pair $(k,\alpha)$ through the relations

$$k = \begin{cases} A \bmod N & \text{if} \quad A \bmod N > 0 \\ N & \text{if} \quad A \bmod N = 0, \end{cases} \qquad \alpha = 1 + \left[\frac{A-1}{N}\right], \tag{4.138}$$

where $[\cdot]$ denotes the integer part (floor function). The matrix $\Gamma$ is defined as the $rN \times rN$ block matrix with entries

$$\Gamma_{kl}^{\alpha\beta} = \begin{cases} \frac{\gamma_k^\alpha}{t_k}\delta_{\alpha\beta} & \text{for} \quad k = l \\ \frac{p_k^{\alpha\,\sharp} q_l^\beta}{t_k - t_l} & \text{for} \quad k \neq l, \end{cases} \tag{4.139}$$

where the upper indices denote the block entry and the lower indices the entries of each block. The matrix $\Gamma$ is symmetric when $p_k^{\alpha\,\sharp} q_l^\beta = -p_l^{\beta\,\sharp} q_k^\alpha$ for $k \neq l$ and all $\alpha, \beta$ in $\{1,2,...,r\}$. Another set of additional assumptions that have been true for all applications we have seen so far is the condition $p_k^{\alpha\,\sharp} q_l^\beta = 0$ for $k \neq l$ and $\alpha \neq \beta$. In this case, the off-diagonal blocks of $\Gamma$ vanish. From (4.137), we find the vectors $r_B$

$$r_B = \sum_{A=1}^{rN} q_A \left(\Gamma^{-1}\right)_{AB}. \tag{4.140}$$

For the vectors $s_k^\alpha$ in (4.132), we require that the product $\left(\mathcal{M}(t,x) A_+^{-1}\right)^\sharp$ have no poles at $t = -\frac{1}{t_k}$. The condition reads

$$p_k^\alpha = t_k^{-1} s_k^\alpha \gamma_k^\alpha + \sum_{l\neq k}^{N} \sum_{\beta=1}^{r} \frac{1}{t_k - t_l} s_l^\beta p_k^{\alpha\,\sharp} q_l^\beta \quad \Longleftrightarrow \quad p_A = \sum_{B=1}^{rN} \Gamma_{AB} s_B \tag{4.141}$$

and the vectors $s_A$ are given by [5]

$$s_A = \sum_{B=1}^{rN} = \left(\Gamma^{-1}\right)_{AB} p_B. \tag{4.142}$$

This completes the solution of the factorisation problem (4.50) and the desired new solution $M$ is given by

$$M = A_+^{-1}(\infty) = \mathbb{1} + \sum_{A,B=1}^{rN} q_A t_A^{-1} \left(\Gamma^{-1}\right)_{AB} p_B^\sharp, \tag{4.143}$$

where $t_A = t_k^\alpha = t_k$ for all values of $\alpha$.

[5]The matrix equation is now $p = s\,\Gamma^T$, where $p$, $s$ are $n \times rN$ matrices whose columns are the vectors $p_k^\alpha$,$s_k^\alpha$ respectively and are defined similarly to matrices $q$ and $r$.

### Conformal factor

The computation of the conformal factor in the multisoliton case with residues of rank $r$ proceeds along the same lines as in section 4.2. From equation (4.45a) follows (4.76) which for $r \geq 1$ gives the following formula for the conformal factor

$$f_E^4 = k_{BM} \cdot \det \Gamma \cdot \prod_{A=1}^{rN} (t_A \nu_A) = k_{BM} \cdot \det \Gamma \cdot \prod_{k=1}^{N} (t_k \nu_k)^r \ . \tag{4.144}$$

We note that, the formula (4.79) for the rank-2 case in section 4.2 is recovered by the above formula, since in that case the $(rN \times rN)$ matrix $(\Gamma_{AB})$ of this section becomes a $4 \times 4$ block diagonal matrix with two repeated blocks $\Gamma_{kl}$ (cf. (4.71)) once the additional assumptions (4.67) are made. This means that $\det(\Gamma_{AB}) = (\det(\Gamma_{kl}))^2$ and with $r = 2$ the formula (4.144) reproduces (4.79).

Moreover, we note that for the case $r = 1$ which was discussed in chapter 3 (Einstein gravity), the formula for the conformal factor takes the form (4.144) when one starting from a Lagrangian of the form (2.50) changes the parameterization of $V$ such that $\langle P_m, P_n \rangle = \mathrm{Tr}(P_m P_n)$ is consistent with the convention (2.38). We have not changed the parameterization there in order to better facilitate comparison with the relevant references in chapter 2 and 3.

# Chapter 5

# Reaching the JMaRT solution

Based on the joint work [64], in this chapter we will continue the discussion on the inverse scattering technique in STU supergravity but take a step further: we will account for solutions with five-dimensional asymptotics, such as the Myers-Perry solution [29] (in particular its Euclidean version) and show that through charging transformations and uplifting to six dimensions, we can reach the smooth JMaRT geometry [30].

The JMaRT geometry is a smooth, non-supersymmetric soliton solution that becomes important in connection with Mathur's fuzzball proposal [65]. This is an attempt to understand the microstate structure of black holes in the framework of string theory. It advocates that the classically obscured regions of a black hole, that is singularities and horizons, are understood at the quantum level as a super-dense state of strings "fused together" into a smooth geometry, which classically appears as a "typical" black hole. In the low energy limit of string theory, namely supergravity, the search and construction of horizonless solutions - that have the same mass and charges as the black hole- has attracted the interest, since one could test whether these solutions could account for the black hole entropy. Non-extremal solutions of this type have been found in [30, 66, 67, 68, 69] among which the JMaRT solution [30] and the running-Bolt solutions [66, 67]. Our motivation for this inverse scattering construction of the JMaRT fuzzball lies in the understanding of this solution from a different point of view, as well as in the possibility that more such solutions could be found in this way.

In the article [64] with A. Kleinschmidt and A. Virmani, we showed that the BM method can be used to reach the JMaRT solution but some modifications in the computational path were needed. These entail a different order of dimensional reductions of the six-dimensional theory (4.2), namely starting from a timelike reduction and subsequently performing the spacelike reductions to obtain the integrable theory (in two dimensions). In this set-up, we start by the construction of the Myers-Perry instanton which is a solution to the five-dimensional Euclidean supergravity that we obtain after the first step of the dimensional reduction. After appropriate charging transformations and six-dimensional uplifting of the Myers-Perry instanton we are able to reconstruct the (singly rotating) fuzzball solution of [30].

Finally, at the end of this chapter, we present for the first time the monodromy matrix for the doubly rotating Myers-Perry black hole [29] within the STU set-up.

## 5.1 Preliminaries

### Dimensional reduction

Since our calculations in this chapter require a different path in terms of the reduction of the six dimensional theory (4.2), we will start by outlining this process and giving a few preliminary details on the structure of the reduced theory in three-dimensions. The latter is again a $\mathrm{SO}(4,4)/(\mathrm{SO}(2,2)\times\mathrm{SO}(2,2))$ coset model. There are a few changes in the set-up (compared to chapter 4) that we will note along the way. The reduction from $D=6$ down to $D=3$ is performed in the following order

$$D=6 \xrightarrow{t} D=5 \xrightarrow{\phi_+} D=4 \xrightarrow{y} D=3, \tag{5.1}$$

i.e., first a time-like reduction to a Euclidean $D=5$ theory and then two space-like reductions[1]. The calculations proceed in the standard way of Kaluza-Klein reductions (for a general reference, see e.g. [37]). For calculations very similar to the ones performed here we refer to [58, 70],[71].

### Time-like reduction to $D=5$

For the reduction over time of (4.2) from $D$=6 to $D$=5, we make the reduction ansatz for the metric

$$ds_6^2 = -e^{\sqrt{\frac{3}{2}}\phi_6}(dt + A^1_{[1]})^2 + e^{-\frac{1}{\sqrt{6}}\phi_6} ds_5^2 \tag{5.2}$$

and write the three-form field in (4.2) as in (4.5). As in section 4.1, the five dimensional theory contains three one-forms $A^I_{[1]}$, $I=1,2,3$; one of them is the one-form $A^1_{[1]}$ from the Kaluza-Klein reduction (5.2) while there is also $A^2_{[1]}$ from the reduction (4.5) and the third one comes from the dualization of the two-form $C_{[2]}$ in (4.5). The duality relation that we find (similarly to section 4.1, cf. (4.7)) reads

$$F^{(5d)}_{[3]} = e^{\sqrt{2}\Phi - \sqrt{\frac{2}{3}}\phi_6} \star_5 F^3_{[2]}, \tag{5.3}$$

where $F^3_{[2]} = dA^3_{[1]}$ and the sign differences compared to (4.5) are due to the reduction over a timelike direction instead of a spacelike. The resulting Euclidean $D=5$ theory can be written as

$$\mathcal{L}^{(5d)} = R^{(5d)} - \frac{1}{2}\star_5 G_{IJ} dh^I \wedge dh^J + \frac{1}{2} G_{IJ} \star_5 F^I_{[2]} \wedge F^J_{[2]} + \frac{1}{6} C_{IJK} F^I_{[2]} \wedge F^J_{[2]} \wedge A^K_{[1]} \tag{5.4}$$

[1] For simplicity we use notation $t, \phi_+$, and $y$, to denote directions over which we perform dimensional reduction. It should be kept in mind that only asymptotically this notation is fully justified.

where we have defined

$$h^1 = e^{-\frac{2}{\sqrt{6}}\phi_6}, \quad h^2 = e^{\frac{1}{\sqrt{2}}\Phi+\frac{1}{\sqrt{6}}\phi_6}, \quad h^3 = e^{-\frac{1}{\sqrt{2}}\Phi+\frac{1}{\sqrt{6}}\phi_6}, \tag{5.5}$$

satisfying $h^1h^2h^3 = 1$. $G_{IJ}$ and $C_{IJK}$ are defined in the same way as in section 4.1. We note that the only difference between Lagrangians (5.4) and (4.9) is in the sign of the kinetic terms for the vector fields.

### Space-like reduction to $D = 4$

The next step is to reduce this theory over a spatial direction to four dimensions. The reduction ansatz is ($z_5 \equiv \phi_+$)

$$ds_5^2 = \check{f}^2(dz_5 + \check{A}^0_{[1]})^2 + \check{f}^{-1}ds_4^2 \tag{5.6}$$

$$A^I_{[1]} = \chi^I \left(dz_5 + \check{A}^0_{[1]}\right) + \check{A}^I_{[1]} \tag{5.7}$$

and the Lagrangian in $D = 4$ takes the form

$$\begin{aligned}\mathcal{L}^{(4d)} &= R^{(4d)} \star_4 1 - \frac{1}{2}G_{IJ} \star_4 dh^I \wedge dh^J - \frac{3}{2}\check{f}^{-2} \star_4 d\check{f} \wedge d\check{f} - \frac{1}{2}\check{f}^3 \star_4 \check{F}^0_{[2]} \wedge \check{F}^0_{[2]} \\ &+ \frac{1}{2}\check{f}^{-2}G_{IJ} \star_4 d\chi^I \wedge d\chi^J + \frac{1}{2}\check{f}G_{IJ} \star_4 \left(\check{F}^I_{[2]} + \chi^I\check{F}^0_{[2]}\right) \wedge \left(\check{F}^J_{[2]} + \chi^J\check{F}^0_{[2]}\right) \\ &+ \frac{1}{2}C_{IJK}\chi^I\check{F}^J_{[2]} \wedge \check{F}^K_{[2]} + \frac{1}{2}C_{IJK}\chi^I\chi^J\check{F}^0_{[2]} \wedge \check{F}^K_{[2]} + \frac{1}{6}C_{IJK}\chi^I\chi^J\chi^K\check{F}^0_{[2]} \wedge \check{F}^0_{[2]}\end{aligned} \tag{5.8}$$

with $\check{F}^\Lambda_{[2]} = d\check{A}^\Lambda_{[1]}$, $\Lambda = 0, 1, 2, 3$. The above Lagrangian describes an $\mathcal{N} = 2$ Euclidean supergravity theory; it is matched to the Euclidean version of (4.14) with prepotential

$$F = -\frac{X^1X^2X^3}{X^0} \tag{5.9}$$

in the gauge $X^0 = 1$. In the Euclidean formalism, the four-dimensional theory looks exactly the same, i.e.

$$\mathcal{L}^{(4d)} = R^{(4d)} \star_4 1 - 2g_{I\bar{J}} \star_4 dX^I d\bar{X}^{\bar{J}} + \frac{1}{2}\check{F}^\Lambda_{[2]} \wedge \check{G}_{\Lambda[2]}, \tag{5.10}$$

but now the complex quantities are defined via the para-imaginary unit $e$, satisfying $e^2 = 1$ and $\bar{e} = -e$ instead of the standard imaginary unit. For more details on $\mathcal{N} = 2$ Euclidean supergravity, see references [72, 73, 74]. The fields $X^I$ are split complex scalars related to the scalars in (5.8) as

$$X^I = -\chi^I - e\check{f}h^I = x^I - ey^I, \tag{5.11}$$

where as before we set $x^I = -\chi^I$ and $y^I = \check{f}h^I$. In this context, the special Kähler geometry is replaced by special para-Kähler geometry where the Kähler metric $g_{I\bar{J}} = \partial_I\partial_{\bar{J}}K$ is derived through the potential

$$K = -\log\left[-e(\bar{X}^\Lambda F_\Lambda - \bar{F}_\Lambda X^\Lambda)\right], \tag{5.12}$$

with $F_\Lambda = \partial_\Lambda F$. Moreover, we define the para-complex symmetric matrix

$$N_{\Lambda\Sigma} = \bar{F}_{\Lambda\Sigma} + 2e\frac{\mathrm{Im}F_{\Lambda K}\,\mathrm{Im}F_{\Sigma P}\,X^K X^P}{\mathrm{Im}F_{MN}\,X^M X^N}, \tag{5.13}$$

and with the above definition the two-forms $\check{G}_{\Lambda[2]}$ in (5.10) are given by

$$\check{G}_{\Lambda[2]} = (\mathrm{Re}N)_{\Lambda\Sigma}\,\check{F}^{\Sigma}_{[2]} + (\mathrm{Im}N)_{\Lambda\Sigma} \star_4 \check{F}^{\Sigma}_{[2]}. \tag{5.14}$$

With the choices made here to write (5.8) in the form (5.10), the real and imaginary part of matrix $N$ read

$$\mathrm{Re}\,N = \begin{pmatrix} -2x_1x_2x_3 & x_2x_3 & x_1x_3 & x_1x_2 \\ x_2x_3 & 0 & -x_3 & -x_2 \\ x_1x_3 & -x_3 & 0 & -x_1 \\ x_1x_2 & -x_2 & -x_1 & 0 \end{pmatrix},$$

$$\mathrm{Im}\,N = \begin{pmatrix} -y_1y_2y_3 + \frac{y_2y_3x_1^2}{y_1} + \frac{y_1y_3x_2^2}{y_2} + \frac{y_1y_2x_3^2}{y_3} & -\frac{x_1y_2y_3}{y_1} & -\frac{x_2y_1y_3}{y_2} & -\frac{x_3y_1y_2}{y_3} \\ -\frac{x_1y_2y_3}{y_1} & \frac{y_2y_3}{y_1} & 0 & 0 \\ -\frac{x_2y_1y_3}{y_2} & 0 & \frac{y_1y_3}{y_2} & 0 \\ -\frac{x_3y_1y_2}{y_3} & 0 & 0 & \frac{y_1y_2}{y_3} \end{pmatrix}. \tag{5.15}$$

### Reduction to $D = 3$

Further reduction brings us to the coset model SO(4, 4)/(SO(2, 2)×SO(2, 2)). There are a few details that differ from the reduction to $D = 3$ in chapter 4, owing to the different order of dimensional reductions that are performed here. In the present case the four-dimensional reduction ansatz reads

$$ds_4^2 = e^{2U}(dy + \omega_3)^2 + e^{-2U}ds_3^2, \tag{5.16}$$

with $ds_3^2$ as given in (4.21). This is now a Euclidean metric, as opposed to the Lorentzian one in (4.21) and in place of the time coordinate is the $y$-coordinate. The one-forms $\check{A}^{\Lambda}_{[1]}$ in four dimensions written in terms of three-dimensional quantities read

$$\check{A}^{\Lambda}_{[1]} = \zeta^\Lambda(dy + \omega_3) + A_3^\Lambda. \tag{5.17}$$

The five one-forms $\omega_3, A_3^\Lambda$ in three-dimensions are dualized into scalars through the relations

$$-d\sigma = -2e^{4U} \star d\omega_3 - \left(\zeta^\Lambda d\tilde{\zeta}_\Lambda + \tilde{\zeta}_\Lambda d\zeta^\Lambda\right) \tag{5.18}$$

$$-d\tilde{\zeta}_\Lambda = e^{2U}(\mathrm{Im}N)_{\Lambda\Sigma} \star \left(dA_3^\Sigma + \zeta^\Sigma d\omega_3\right) + (\mathrm{Re}N)_{\Lambda\Sigma}d\zeta^\Sigma. \tag{5.19}$$

With the above dualizations, we are lead to the reduced three-dimensional theory that has the form of a $\sigma$-model

$$\mathcal{L}^{(3d)} = R^{(3d)} \star_3 1 - \frac{1}{2}h_{ij}\partial\varphi^i\partial\varphi^j, \tag{5.20}$$

whose target manifold is $\mathrm{G}_E/\mathrm{K}_E = \mathrm{SO}(4,4)/(\mathrm{SO}(2,2)\times\mathrm{SO}(2,2))$ with coordinates the 16 scalars $\varphi^i = \Big(U, z^I, \bar{z}^I, \zeta^\Lambda, \tilde{\zeta}_\Lambda, \sigma\Big)$ of signature $(8,8)$. The metric on the coset manifold reads

$$
\begin{aligned}
&h_{ij}d\varphi^i d\varphi^j = 4dU^2 + 4g_{I\bar{J}}dz^I d\bar{z}^{\bar{J}} - \frac{1}{4}e^{-4U}\left(d\sigma + \tilde{\zeta}_\Lambda d\zeta^\Lambda - \zeta^\Lambda d\tilde{\zeta}_\Lambda\right)^2 \\
&+ e^{-2U}\Big(-(\mathrm{Im}N)_{\Lambda\Sigma}d\zeta^\Lambda d\zeta^\Sigma + ((\mathrm{Im}N)^{-1})^{\Lambda\Sigma}\left(d\tilde{\zeta}_\Lambda + (\mathrm{Re}N)_{\Lambda P}d\zeta^P\right) \\
&\left(d\tilde{\zeta}_\Sigma + (\mathrm{Re}N)_{\Sigma P}d\zeta^P\right)\Big),
\end{aligned}
\tag{5.21}
$$

with $z^I = x^I - ey^I$ and we note again that the sign changes with respect to (4.26) are due to the different order of dimensional reductions. Using again the $\mathrm{SO}(4,4)$ basis as in chapter 4 (also in [61, 58, 51]), the preserved metric is given by (4.34). The symmetric space automorphism $\tilde{\tau}$ that fixes the denominator group according to

$$
\mathrm{K}_E = \mathrm{SO}(2,2)\times\mathrm{SO}(2,2) = \left\{g\in\mathrm{SO}(4,4)\,|\, g^T\eta' g = \eta'\right\} \tag{5.22}
$$

is defined explicitly by

$$
\begin{aligned}
&\tilde{\tau}(H_0) = -H_0, && \tilde{\tau}(H_I) = -H_I, && (5.23a)\\
&\tilde{\tau}(E_0) = +F_0, && \tilde{\tau}(E_I) = +F_I, && (5.23b)\\
&\tilde{\tau}(E_{q_0}) = -F_{q_0}, && \tilde{\tau}(E_{q_I}) = +F_{q_I}, && (5.23c)\\
&\tilde{\tau}(E_{p^0}) = +F_{p^0}, && \tilde{\tau}(E_{p^I}) = -F_{p^I} && (5.23d)
\end{aligned}
$$

and the anti-involution "$\sharp$" on a $\mathfrak{so}(4,4)$ Lie algebra element $x$ is given by

$$
x^\sharp = -\tilde{\tau}(x) = \eta' x^T \eta', \tag{5.24}
$$

with the invariant metric (signs change compared to (4.31))

$$
\eta' = \mathrm{diag}(+,-,-,+,+,-,-,+). \tag{5.25}
$$

For the coset element $V(x)$, we will use the same parameterization as in chapter 4, namely (4.32) and write the coset metric as

$$
h_{ij}d\varphi^i d\varphi^j = \mathrm{Tr}(PP) \tag{5.26}
$$

with $P = \frac{1}{2}\left(dVV^{-1} + (dVV^{-1})^\sharp\right)$.

## 5.2 Riemann-Hilbert factorisation for $\mathrm{SO}(4,4)$ and asymptotically flat solutions in $D=5$

The main adjustment in applying the BM method to capture solutions with flat five-dimensional asymptotics is to determine the right asymptotic behaviour $\mathcal{M}(\infty)$ for the monodromy matrix. In this respect, the discussion in [63] on stationary axisymmetric solutions of five-dimensional gravity was very useful. The Riemann-Hilbert factorisation of section 4.2 remains the same, with the only modification being the asymptotic behaviour of $\mathcal{M}(w)$.

We now take $\mathcal{M}(w)$ such that

$$\mathcal{M}(\infty) = Y, \tag{5.27}$$

where $Y$ is a constant matrix whose form is determined by the asymptotics of the solution (we will give more details on this shortly) and satisfies $Y^\sharp = Y$. The solitonic ansatz for $\mathcal{M}(w)$ reads

$$\mathcal{M}(w) = Y + \sum_{k=1}^{N} \frac{A_k}{w - w_k}, \tag{5.28a}$$

$$\mathcal{M}^{-1}(w) = \eta \mathcal{M}^T \eta = \eta \left( Y + \sum_{k=1}^{N} \frac{A_k^T}{w - w_k} \right) \eta, \tag{5.28b}$$

where the only change with respect to (4.51) is the first term in the expansion. The factorization algorithm proceeds exactly the same as in section 4.2, with the same parameterization of the residues and the same conditions on the vectors $a_k, b_k$. The matrix $A_+$ is given by (4.60) and the matrices $C_k$ by (4.61). The matrix $\Gamma$ is defined again by (4.71). The final result is the matrix $M(x)$ in the factorisation (4.50), which is now given by

$$M(x) = Y A_+^{-1}(\infty, x), \tag{5.29}$$

as a result of (5.27),(4.50) and $A_+(0,x) = \mathbb{1} = A_-(\infty, x)$. Finally, the conformal factor $f_E$ in the two-dimensional base metric $ds_2^2 = f_E^2(d\rho^2 + dz^2)$ is determined by the formula (4.79).

## Asymptotic behaviour of $M(x)$, $\mathcal{M}(w)$

We will now examine the asymptotic behaviour of the coset metric $M(x)$ as well as the monodromy matrix $\mathcal{M}(w)$ for $D = 5$ asymptotically flat solutions. First, let us consider five-dimensional Minkowski space that is trivially uplifted to six dimensions along the $y$-direction, with line element

$$ds^2 = -dt^2 + dy^2 + dr^2 + r^2 \left[ d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2 \right], \tag{5.30}$$

where $\theta \in [0, \frac{\pi}{2}]$ , $(\phi, \psi)$ are standard angular coordinates with range $[0, 2\pi)$ and $y$ is a periodic coordinate around a circle. Following [63], we change to the coordinates[2]

$$\phi_+ = \frac{1}{2}(\psi + \phi), \quad \phi_- = (\phi - \psi) \tag{5.31}$$

[2]The specific normalization for these coordinates is chosen to simplify later expressions.

and obtain the metric

$$ds^2 = -dt^2 + dy^2 + dr^2 + r^2 \left[ d\theta^2 + \frac{1}{4} d\phi_-^2 + d\phi_+^2 - \cos 2\theta d\phi_- d\phi_+ \right] . \qquad (5.32)$$

The reason we choose to change coordinates in this way is so that the coset matrices asymptotically tend to constant values (see discussion in [63],[75]). Working with standard angular coordinates, leads to infinities in the asymptotic behaviour of the solution, which would in turn require us to include poles at infinity in the ansatz (5.28a). As it is not yet clear how to incorporate this kind of poles in the formalism presented here, we choose to work with the "nicer" coordinates (5.31).

Now if we wish to reach (5.32) as a solution of the STU coset model, it corresponds to the following profiles of the fields in three dimensions :

$$\begin{aligned} e^{2U} &= r, \quad y^1 = y^2 = y^3 = r, \quad \tilde{\zeta}_0 = r^2, & (5.33) \\ A^0 &= -\frac{1}{2}\cos 2\theta d\phi_- , & (5.34) \\ ds_3^2 &= r^2 \left[ dr^2 + r^2 d\theta^2 + r^2 \cos^2\theta \sin^2\theta d\phi_-^2 \right] , & (5.35) \end{aligned}$$

while the rest of the fields vanish. Clearly, in contrast to flat space in four dimensions, we get some non-trivial expressions for the three-dimensional scalar fields and one-forms. With the parameterization (4.32), the relation $M = V^\sharp V$ and the non-zero scalars given in (5.33) we get the matrix $M(x)$

$$M(x) = \begin{pmatrix} \frac{1}{r^2} & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} , \qquad (5.36)$$

which in the limit $r \to \infty$ takes on the constant value

$$Y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \qquad (5.37)$$

For solutions that are asymptotically flat in five dimensions, we therefore require that the monodromy matrix $\mathcal{M}(w)$ asymptotes to $Y$ as $w \to \infty$ as shown in the ansatz (5.28a).

### Charging transformations

Starting with a monodromy matrix $\mathcal{M}(w)$ whose asymptotic behaviour is determined by $Y$ above, applying a charging transformation of the type

$$\mathcal{M}_{\text{charged}}(w) = g^{\sharp}\mathcal{M}_{\text{seed}}(w)g, \tag{5.38}$$

with $g$ a $w$-independent transformation in $\mathrm{SO}(4,4)$, must be such that it preserves the asymptotics, i.e.

$$\mathcal{M}_{\text{charged}}(w \to \infty) = Y. \tag{5.39}$$

Thus we need to find the subgroup of elements $g_D \in \mathrm{SO}(4,4)$ such that

$$g_D^{\sharp} Y g_D = Y. \tag{5.40}$$

From the observation that there is an $SO(4,4)$, "$\sharp$"-invariant matrix that satisfies

$$D^{\sharp} D = Y \tag{5.41}$$

with

$$D = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{pmatrix}, \tag{5.42}$$

we deduce that the appropriate charging element must be of the form

$$g_D = D^{-1} k D, \qquad k \in \mathrm{K}_E = \mathrm{SO}(2,2) \times \mathrm{SO}(2,2). \tag{5.43}$$

(Recall that the charging transformations (4.91) preserving the four-dimensional asymptotics are elements in $\mathrm{K}_E$, see section 4.2.1.) The solutions to (5.40) form an $\mathrm{SO}(2,2) \times \mathrm{SO}(2,2)$ subgroup conjugate to $\mathrm{K}_E$ that acts on the monodromy matrix as

$$\mathcal{M}^{g_D}(w) := g_D^{\sharp}(w)\mathcal{M}(w)g_D(w), \tag{5.44}$$

preserving the form (5.28a) and therefore the five-dimensional asymptotics. See also [63, 75] for a similar discussion in the $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(2,1)$ case.

## 5.3 Supergravity configuration

In this section we will explain the reconstruction of the JMaRT fuzzball [30] through the BM inverse scattering method. At first, one may think that starting with a two-soliton ansatz (as in the case of black hole solutions seen before) and constructing the appropriately charged, five-dimensional soliton, the JMaRT solution could then be obtained by trivially uplifting this solution to six dimensions. However, the fact that the range of parameters for which the JMaRT geometry is a smooth and horizonless one is the over-rotating range, (i.e. $M < a^2$) proved to be essential in choosing the right solution to charge-up in five dimensions.

It turned out that constructing the Myers-Perry black hole with the BM method and then applying charging transformations to obtain the five-dimensional Cvetic-Youm solution, was not a suitable path to yield a fuzzball solution in six dimensions. This strategy would require us to consider complex poles to account for "over-rotation"; this types of poles however, lead to solutions with naked singularities [11],[49] and for this reason, the inverse scattering method that we use here is adapted to real poles.

In order to bypass this difficulty of reconciling over-rotating regime with real poles in our solutions, we choose the following approach. By reducing the six dimensional theory over time first (as described earlier in section 5.1), we subsequently obtain a Euclidean theory in five and then four dimensions. Considering the analogy to the Kerr black hole, as a solution to four-dimensional, *Lorentzian* vacuum gravity, it is known that real poles yield an under-rotating solution, while complex conjugate poles lead to naked singularities in the solution. However, in *Euclidean* four-dimensional vacuum gravity, the corresponding solution with real poles is the Kerr *instanton.* We found that in five dimensions, we have the same picture for the Myers-Perry solution. Starting from a two-soliton ansatz and using the BM method as modified in section 5.2, we construct a Euclidean five dimensional soliton solution that turns out to be the Myers-Perry instanton. Adding charges to this solution and trivially lifting it to six dimensions to include a time direction, we are able to match it to the (singly rotating) JMaRT fuzzball.

Let us summarize the steps of our computation and next give the details of the inverse scattering construction.

1. We aim to construct a metric of the form
$$ds_6^2 = -dt^2 + ds_5^2, \tag{5.45}$$
where $ds_5^2$ is the line element of the Euclidean five-dimensional gravity configuration discussed above.

2. On this configuration we apply an SO(4,4) charging transformation of the form (5.43)
$$\mathcal{M}_{\text{new}}(w) = g_D^\sharp \mathcal{M}_{\text{old}}(w) g_D, \tag{5.46}$$
in order to add electric charges.

3. Then we analyse degeneration properties of various Killing vectors and relate the final configuration to the JMaRT fuzzball.

The first of the above steps, starts with the BM construction of a soliton solution with five dimensional asymptotics. We consider the ansatz with two real poles at $w = \pm c$ for the monodromy matrix:

$$\mathcal{M}(w) = Y + \frac{A_1}{w-c} + \frac{A_2}{w+c}, \tag{5.47}$$

where $\mathcal{M}(w) \in \mathrm{SO}(4,4)$ and the residue matrices $A_1$ and $A_2$ are parameterized as [3]

$$A_1 = \alpha_1 a_1 a_1^T \eta' - \beta_1 (\eta b_1)(\eta b_1)^T \eta', \tag{5.48a}$$
$$A_2 = \alpha_2 a_2 a_2^T \eta' - \beta_2 (\eta b_2)(\eta b_2)^T \eta'. \tag{5.48b}$$

Next, we choose the vectors $a_1, a_2$ by inspecting the general form of the matrix $M(x)$ that corresponds to the five dimensional vacuum gravity configuration that we are interested in. A suitable choice is

$$a_1 = \{1, 0, 0, \zeta_{12}, 0, 0, \zeta_{11}, 0\}, \tag{5.49a}$$
$$a_2 = \{\zeta_{21}, 0, 0, 1, 0, 0, \zeta_{22}, 0\}. \tag{5.49b}$$

As in section 4.2.1, the $b_1, b_2$ vectors can be constructed through the matrix $\xi$ that is now given by

$$\xi = a^T \eta' Y^{-1} a = \begin{pmatrix} a_1^T \eta' Y^{-1} a_1 & a_1^T \eta' Y^{-1} a_2 \\ a_2^T \eta' Y^{-1} a_1 & a_2^T \eta' Y^{-1} a_2 \end{pmatrix}, \tag{5.50}$$

where $a$ is the $8 \times 2$ matrix with columns the vectors $a_1, a_2$. Moreover, the matrix $\xi$ is symmetric since $(\eta' Y^{-1})^T = \eta' Y^{-1}$. The $b$-vectors take the form

$$b = (\det \xi) \eta' Y^{-1} a \xi^{-1} \epsilon, \qquad \text{with} \quad \epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{5.51}$$

and we have set the constant parameters $\alpha_k, \beta_k$ as

$$\alpha_1 = \frac{2c}{\det \xi} \xi_{22}, \qquad \alpha_2 = -\frac{2c}{\det \xi} \xi_{11}, \tag{5.52a}$$
$$\beta_1 = -\frac{1}{\det \xi} \alpha_1, \qquad \beta_2 = -\frac{1}{\det \xi} \alpha_2. \tag{5.52b}$$

The above choices for the $a_k, b_k$ vectors and the $\alpha_k, \beta_k$ parameters are such that the monodromy matrix satisfies all coset constraints. Following the factorization algorithm of section 4.2, we reach a matrix $M(x)$ from which we can read off the scalars. For this spacetime configuration, the only non-vanishing fields are

$$U, \;\; y^I = y, \;\; \zeta^0, \;\; \tilde{\zeta}_0 \tag{5.53}$$

[3] Recall that $\eta'$ is now given by (5.25).

where all $y^I$ fields are equal. The rest of the fields $(x^I, \zeta^I, \tilde{\zeta}_I)$ are zero. Now denoting the entries of the matrix $M(x)$ as $m_{ab}$, we express the fields (5.53) as[4]

$$y = \sqrt{\frac{m_{44}}{m_{33}}}, \qquad e^{2U} = \frac{1}{\sqrt{m_{44}m_{33}}}, \tag{5.55a}$$

$$\zeta_0 = -\frac{m_{41}}{m_{44}}, \qquad \tilde{\zeta}^0 = \frac{m_{35}}{m_{33}}, \tag{5.55b}$$

and $\sigma$ takes value

$$\sigma = -\frac{m_{35}m_{41} + 2m_{33}m_{47}}{m_{33}m_{44}}. \tag{5.56}$$

We can simplify the explicit profiles from the above expressions, by setting

$$\zeta_{12} = 0, \quad \zeta_{21} = 0 \tag{5.57}$$

and thus end up with two parameters, $\zeta_{11}, \zeta_{22}$, enough to describe a singly rotating solution. This will allow for a simpler calculation while at the same time capturing enough structure for the solution we wish to reach. (In principle, keeping all parameters non-zero, would eventually lead to the doubly rotating JMaRT fuzzball. However, this a very computationally demanding calculation that we will not pursue here).

The result of the Riemann-Hilbert factorization, that is the matrix $M(x)$, is a function of the variables $t_1, t_2$ (the pole locations). To move on with our calculations, we change to the more convenient prolate spherical coordinates $(u, v)$ using relations (3.95). The non-vanishing distinct components of $M$ read

$$m_{33} = -\frac{2\left(2\zeta_{11}(u+1) - \zeta_{22}^2(u+v)\right)}{\left(\zeta_{22}^2 - 2\zeta_{11}\right)\left(2\zeta_{11}\left(1-u^2\right) + \zeta_{22}^2\left(u^2-v^2\right)\right)}, \tag{5.58a}$$

$$m_{44} = \frac{\zeta_{11}^2\left(4u^2+8u+4\right) + 2\zeta_{11}\zeta_{22}^2\left(-2u^2-2u+v^2+2v+1\right) + \zeta_{22}^4\left(u^2-v^2\right)}{\left(\zeta_{22}^2 - 2\zeta_{11}\right)\left(2\zeta_{11}\left(1-u^2\right) + \zeta_{22}^2\left(u^2-v^2\right)\right)}, \tag{5.58b}$$

$$\begin{aligned} m_{35} = \Big(&-2\zeta_{11}\zeta_{22}^2 + u^2\left(\zeta_{22}^2 - 2\zeta_{11}\right)^2 - 2u\left(\zeta_{11}\zeta_{22}^2 - 2\zeta_{11}^2\right) - \zeta_{22}^2v^2\left(\zeta_{22}^2 - 2\zeta_{11}\right) \\ &+ 2\zeta_{11}\zeta_{22}^2v\Big) / \left(\left(\zeta_{22}^2 - 2\zeta_{11}\right)\left(2\zeta_{11}\left(1-u^2\right) + \zeta_{22}^2\left(u^2-v^2\right)\right)\right), \end{aligned} \tag{5.58c}$$

$$m_{47} = -\frac{2\left(2\zeta_{11}^2\zeta_{22} + u\left(4\zeta_{11}^2\zeta_{22} - 2\zeta_{11}\zeta_{22}^3\right) + v\left(2\zeta_{11}\zeta_{22}^3 - 2\zeta_{11}^2\zeta_{22}\right)\right)}{\left(\zeta_{22}^2 - 2\zeta_{11}\right)\left(2\zeta_{11}\left(1-u^2\right) + \zeta_{22}^2\left(u^2-v^2\right)\right)}, \tag{5.58d}$$

$$m_{41} = \frac{2(-2\zeta_{11}\zeta_{22} - 2\zeta_{11}\zeta_{22}v)}{\left(\zeta_{22}^2 - 2\zeta_{11}\right)\left(2\zeta_{11}\left(1-u^2\right) + \zeta_{22}^2\left(u^2-v^2\right)\right)}. \tag{5.58e}$$

[4] For this configuration, the various matrices involved in the calculation acquire a simple form, for instance the imaginary part of matrix $N_{\Lambda\Sigma}$ reads

$$\operatorname{Im} N = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{m_{44}}{m_{33}} & 0 & 0 \\ 0 & 0 & \frac{m_{44}}{m_{33}} & 0 \\ 0 & 0 & 0 & \frac{m_{44}}{m_{33}} \end{pmatrix}. \tag{5.54}$$

From the above entries and (5.55) , we get the explicit expressions for the scalars. We then integrate (5.18),(5.19) to get the one-forms $\omega_3, A_3^\Lambda$

$$\omega_3 = -\frac{2c\zeta_{22}\left(u^2\left(\zeta_{22}^2-2\zeta_{11}\right)+2\zeta_{11}u\left(v^2-1\right)+v^2\left(2\zeta_{11}-\zeta_{22}^2\right)\right)}{\left(2\zeta_{11}-\zeta_{22}^2\right)\left(2\zeta_{11}\left(u^2-1\right)+\zeta_{22}^2\left(v^2-u^2\right)\right)}dz_3, \tag{5.59}$$

$$A_3^\Lambda = 0 \tag{5.60}$$

where $z_3 \equiv \phi_-$. With the above data, we are able to write the three-dimensional base metric

$$\begin{aligned} ds_3^2 \;\; = \;\; & \frac{\left(\zeta_{22}^2-2\zeta_{11}\right)}{16}\left((u^2-v^2)\zeta_{22}^2-2\left(u^2-1\right)\zeta_{11}\right)\left[\frac{du^2}{(u^2-1)}+\frac{dv^2}{(1-v^2)}\right] \\ & +c^2\left(u^2-1\right)\left(1-v^2\right)dz_3^2, \end{aligned} \tag{5.61}$$

where the BM constant factor in (4.79) is fixed such that asymptotic flatness is ensured.

The next step in this calculation is to add charges to the solution. We act on the monodromy matrix according to (5.46), with a charging element of the type (5.43). The element we use is a constant transformation that adds two charges, parameterized by the angles $\delta_2, \delta_3$ :

$$g = \exp\left[\left(i\frac{\pi}{2}-\delta_2\right)K_{q_2}\right]\cdot\exp\left[\left(i\frac{\pi}{2}-\delta_3\right)K_{q_3}\right] \tag{5.62}$$

where $K_{q_I}$, $I = 1,2,3$ are SO(2,2)×SO(2,2) elements defined in (4.43),(4.39),(4.40). We note that the shift $i\frac{\pi}{2}$ in (5.62) is not necessary but it is convenient in that it corresponds to the parameterization used in [30] and thus allows us to match the solution directly to the (singly rotating) JMaRT solution. From the coset model point of view, the shift $i\frac{\pi}{2}$ is associated to a conjugation relation between the distinct $SL(3,\mathbb{R})$ subgroups that correspond to the Euclidean and Lorentzian five dimensional vacuum gravity sector. We will give a note on this at the end of this chapter.

After the transformation (5.46), we factorize the new monodromy matrix and obtain the coset metric $M_{\text{new}}(x)$. We read off the expressions for the scalars which turn out to be quite involved. For reasons of space and readability we present these results in terms of the fields (5.55) which can in turn be explicitly obtained via (5.58). The field profiles (5.55) we denote by the subscript "(MP)". We will also use the shorter notation $c_{2,3} = \cosh\delta_{2,3}$ and $s_{2,3} = \sinh\delta_{2,3}$. For the fields $x_I, y_I$ of the charged solution we get

$$x_1 = \frac{1}{2}c_2c_3\left(\zeta^0_{(\mathrm{MP})}\tilde{\zeta}_{0(\mathrm{MP})} + \sigma_{(\mathrm{MP})}\right), \tag{5.63}$$

$$x_2 = \frac{c_2s_3(y_{(\mathrm{MP})})^3\zeta^0_{(\mathrm{MP})}}{c_2^2e^{2U}_{(\mathrm{MP})} - s_2^2y_{(\mathrm{MP})} + c_2^2(y_{(\mathrm{MP})})^3(\zeta^0_{(\mathrm{MP})})^2}, \tag{5.64}$$

$$x_3 = \frac{s_2c_3(y_{(\mathrm{MP})})^3\zeta^0_{(\mathrm{MP})}}{c_3^2e^{2U}_{(\mathrm{MP})} - s_3^2y_{(\mathrm{MP})} + c_3^2(y_{(\mathrm{MP})})^3(\zeta^0_{(\mathrm{MP})})^2}, \tag{5.65}$$

$$\begin{aligned} y_1 = \Big(& s_2^2y_{(\mathrm{MP})}\left(s_3^2y_{(\mathrm{MP})} - c_3^2e^{2U}_{(\mathrm{MP})}\right) + c_2^2e^{2U}_{(\mathrm{MP})}\Big(c_3^2\left(e^{2U}_{(\mathrm{MP})} + (y_{(\mathrm{MP})})^3(\zeta^0_{(\mathrm{MP})})^2\right) \\ & - s_3^2y_{(\mathrm{MP})}\Big)\Big)^{1/2}, \end{aligned} \tag{5.66}$$

$$\begin{aligned} y_2 = \Big(& y_{(\mathrm{MP})}(e^{4U}_{(\mathrm{MP})}c_2^2c_3^2 + s_2^2s_3^2(y_{(\mathrm{MP})})^2e^{2U}_{(\mathrm{MP})}y_{(\mathrm{MP})}((y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2 + \\ & + s_3^2((y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2 - 1) + s_2^2((y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2 \\ & + s_3^2(-2 + (y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2) - 1)))^{1/2}\Big)\Big/\Big(c_2^2e^{2U}_{(\mathrm{MP})} + (y_{(\mathrm{MP})})^3(\zeta^0_{(\mathrm{MP})})^2 \\ & + s_2^2y_{(\mathrm{MP})}((y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2 - 1)\Big), \end{aligned} \tag{5.67}$$

$$\begin{aligned} y_3 = y_{(\mathrm{MP})}\Big(& e^{4U}_{(\mathrm{MP})}c_2^2c_3^2 + s_2^2s_3^2(y_{(\mathrm{MP})})^2 + e^{2U}_{(\mathrm{MP})}y_{(\mathrm{MP})}((y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2 \\ & + s_3^2((y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2 - 1) + s_2^2((y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2 \\ & + s_3^2((y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2 - 2) - 1))\Big)^{1/2}\Big/ \\ & \Big(c_3^2e^{2U}_{(\mathrm{MP})} + (y_{(\mathrm{MP})})^3(\zeta^0_{(\mathrm{MP})})^2 + s_3^2y_{(\mathrm{MP})}((y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2 - 1)\Big). \end{aligned} \tag{5.68}$$

For the fields $U, \sigma$ we find

$$\begin{aligned} e^{2U} = e^{2U}_{(\mathrm{MP})}y_{(\mathrm{MP})}\Big(& 1/\big(s_2^2y_{(\mathrm{MP})}(-c_3^2e^{2U}_{(\mathrm{MP})} + s_3^2y_{(\mathrm{MP})}) + c_2^2e^{2U}(-s_3^2y_{(\mathrm{MP})} + \\ & + c_3^2(e^{2U}_{(\mathrm{MP})} + (y_{(\mathrm{MP})})^3(\zeta^0_{(\mathrm{MP})})^2)))\Big)^{1/2} \end{aligned} \tag{5.69}$$

$$\begin{aligned} \sigma = \Big(& s_2s_3y_{(\mathrm{MP})}((2 + s_2^2 + s_3^2)y_{(\mathrm{MP})}\zeta^0_{(\mathrm{MP})}\tilde{\zeta}_{0(\mathrm{MP})} + (s_2^2 + s_3^2)y_{(\mathrm{MP})}\sigma_{(\mathrm{MP})} + \\ & - e^{2U}_{(\mathrm{MP})}(2 + s_2^2 + s_3^2)(\zeta^0_{(\mathrm{MP})}\tilde{\zeta}_{0(\mathrm{MP})} + \sigma_{(\mathrm{MP})}))\Big)\Big/ \\ & \Big(2(e^{4U}_{(\mathrm{MP})}c_2^2c_3^2 + s_2^2s_3^2(y_{(\mathrm{MP})})^2 + e^{2U}_{(\mathrm{MP})}y_{(\mathrm{MP})}(-s_2^2 - s_3^2 - 2s_2^2s_3^2 \\ & + c_2^2c_3^2(y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2))\Big). \end{aligned} \tag{5.70}$$

The rest of the scalars are $\zeta^\Lambda$ and $\tilde{\zeta}_\Lambda$ which read

$$\begin{aligned} \zeta^0 = \Big(& -s_2s_3(y_{(\mathrm{MP})})^2\zeta^0_{(\mathrm{MP})}\Big)\Big/\Big(s_2^2y_{(\mathrm{MP})}(s_3^2y_{(\mathrm{MP})} - c_3^2e^{2U}_{(\mathrm{MP})}) \\ & + c_2^2e^{2U}_{(\mathrm{MP})}(c_3^2(e^{2U}_{(\mathrm{MP})} + (y_{(\mathrm{MP})})^3(\zeta^0_{(\mathrm{MP})})^2) - s_3^2y_{(\mathrm{MP})})\Big), \end{aligned} \tag{5.71}$$

$$
\begin{aligned}
\zeta^1 = &-\Big(c_2c_3s_2s_3(y_{(\mathrm{MP})})^2\zeta^0_{(\mathrm{MP})}\Big(\zeta^0_{(\mathrm{MP})}\tilde{\zeta}_{0(\mathrm{MP})}+\sigma_{(\mathrm{MP})}\Big)\Big)\Big/\\
&\Big(2\Big(s_2^2y_{(\mathrm{MP})}(s_3^2y_{(\mathrm{MP})}-c_3^2e^{2U}_{(\mathrm{MP})})+c_2^2e^{2U}_{(\mathrm{MP})}(c_3^2(e^{2U}_{(\mathrm{MP})}\\
&+(y_{(\mathrm{MP})})^3(\zeta^0_{(\mathrm{MP})})^2)-s_3^2y_{(\mathrm{MP})})\Big)\Big), \qquad (5.72)\\
\zeta^2 = &-\Big(c_2s_2(-s_3^2(e^{2U}_{(\mathrm{MP})}-y_{(\mathrm{MP})})y_{(\mathrm{MP})}+c_3^2e^{2U}_{(\mathrm{MP})}(e^{2U}_{(\mathrm{MP})}-y_{(\mathrm{MP})}\\
&+(y_{(\mathrm{MP})})^3(\zeta^0_{(\mathrm{MP})})^2))\Big)\Big/\Big(e^{4U}_{(\mathrm{MP})}c_2^2c_3^2+s_2^2s_3^2(y_{(\mathrm{MP})})^2+e^{2U}_{(\mathrm{MP})}y_{(\mathrm{MP})}\\
&((y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2+s_3^2((y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2-1)+s_2^2(-1+(y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2\\
&+s_3^2((y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2)-2))\Big), \qquad (5.73)\\
\zeta^3 = &-\Big(c_3s_3(-s_2^2(e^{2U}_{(\mathrm{MP})}-y_{(\mathrm{MP})})y_{(\mathrm{MP})}+c_2^2e^{2U}_{(\mathrm{MP})}(e^{2U}_{(\mathrm{MP})}-y_{(\mathrm{MP})}\\
&+(y_{(\mathrm{MP})})^3(\zeta^0_{(\mathrm{MP})})^2))\Big)\Big/\Big(e^{4U}_{(\mathrm{MP})}c_2^2c_3^2+s_2^2s_3^2(y_{(\mathrm{MP})})^2+e^{2U}_{(\mathrm{MP})}y_{(\mathrm{MP})}\\
&((y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2+s_3^2((y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2-1)+s_2^2(-1+(y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2\\
&+s_3^2(-2+(y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2)))\Big) \qquad (5.74)
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{\zeta}_0 = &\Big(2e^{4U}_{(\mathrm{MP})}c_2^2c_3^2\tilde{\zeta}_{0(\mathrm{MP})}+2s_2^2s_3^2(y_{(\mathrm{MP})})^2\tilde{\zeta}_{0(\mathrm{MP})}+e^{2U}_{(\mathrm{MP})}y_{(\mathrm{MP})}((y_{(\mathrm{MP})})^2\zeta^0_{(\mathrm{MP})}\\
&(\zeta^0_{(\mathrm{MP})}\tilde{\zeta}_{0(\mathrm{MP})}-\sigma_{(\mathrm{MP})})+s_3^2(-2\tilde{\zeta}_{0(\mathrm{MP})}+(y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2\tilde{\zeta}_{0(\mathrm{MP})}\\
&-(y_{(\mathrm{MP})})^2\zeta^0_{(\mathrm{MP})}\sigma_{(\mathrm{MP})})+s_2^2((-2+(y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2\\
&+s_3^2(-4+(y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2))\tilde{\zeta}_{0(\mathrm{MP})}-c_3^2(y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2\sigma_{(\mathrm{MP})}))\Big)\Big/\\
&\Big(2\big(e^{4U}_{(\mathrm{MP})}c_2^2c_3^2+s_2^2s_3^2(y_{(\mathrm{MP})})^2+e^{2U}_{(\mathrm{MP})}y_{(\mathrm{MP})}((y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2\\
&+s_3^2(-1+(y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2)+s_2^2(-1+(y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2\\
&+s_3^2(-2+(y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2))))\Big), \qquad (5.75)\\
\tilde{\zeta}_1 = &\Big(c_2c_3e^{2U}_{(\mathrm{MP})}(y_{(\mathrm{MP})})^3\zeta^0_{(\mathrm{MP})}\Big)\Big/\Big(e^{4U}_{(\mathrm{MP})}c_2^2c_3^2+s_2^2s_3^2(y_{(\mathrm{MP})})^2+e^{2U}_{(\mathrm{MP})}y_{(\mathrm{MP})}\\
&((y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2+s_3^2(-1+(y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2)+s_2^2(-1+(y_{(\mathrm{MP})})^2\\
&(\zeta^0_{(\mathrm{MP})})^2+s_3^2(-2+(y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2)))\Big), \qquad (5.76)
\end{aligned}
$$

$$\tilde{\zeta}_2 = \Big(c_2 s_3 y_{(\mathrm{MP})}(e^{2U}_{(\mathrm{MP})}c_2^2 - s_2^2 y_{(\mathrm{MP})})(\zeta^0_{(\mathrm{MP})}\tilde{\zeta}_{0(\mathrm{MP})} + \sigma_{(\mathrm{MP})})\Big)\Big/$$
$$\Big(2(e^{4U}_{(\mathrm{MP})}c_2^2c_3^2 + s_2^2s_3^2(y_{(\mathrm{MP})})^2 + e^{2U}_{(\mathrm{MP})}y_{(\mathrm{MP})}(-s_2^2 - s_3^2 - 2s_2^2s_3^2$$
$$+ c_2^2c_3^2(y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2))\Big), \tag{5.77}$$
$$\tilde{\zeta}_3 = \Big(s_2 c_3 y_{(\mathrm{MP})}(e^{2U}_{(\mathrm{MP})}c_3^2 - s_3^2 y_{(\mathrm{MP})})(\zeta^0_{(\mathrm{MP})}\tilde{\zeta}_{0(\mathrm{MP})} + \sigma_{(\mathrm{MP})}))\Big/$$
$$\Big(2(e^{4U}_{(\mathrm{MP})}c_2^2c_3^2 + s_2^2s_3^2(y_{(\mathrm{MP})})^2 + e^{2U}_{(\mathrm{MP})}y_{(\mathrm{MP})}(-s_2^2 - s_3^2 - 2s_2^2s_3^2$$
$$+ c_2^2c_3^2(y_{(\mathrm{MP})})^2(\zeta^0_{(\mathrm{MP})})^2))\Big). \tag{5.78}$$

In order to write the spacetime metric, we require the one-forms $\omega_3$ and $A_3^\Lambda$ that are dual to the scalars $\sigma, \tilde{\zeta}_\Lambda$. After integration of the corresponding duality relations, we get ($z_3 \equiv \phi_-$)

$$\omega_3 = \frac{2c\zeta_{22}s_2s_3\left(u^2\left(\zeta_{22}^2 - 2\zeta_{11}\right) + 2\zeta_{11}u\left(v^2-1\right) + v^2\left(2\zeta_{11} - \zeta_{22}^2\right)\right)}{\left(2\zeta_{11} - \zeta_{22}^2\right)\left(2\zeta_{11}\left(u^2-1\right) + \zeta_{22}^2\left(v^2-u^2\right)\right)}dz_3, \tag{5.79}$$
$$A_3^0 = \frac{2c\left(-\left(u^2-1\right)v\left(2\zeta_{11} - \zeta_{22}^2\right) - \zeta_{22}^2 u + \zeta_{22}^2 uv^2\right)}{\left(2\zeta_{11} - \zeta_{22}^2\right)\left(2\zeta_{11}\left(u^2-1\right) + \zeta_{22}^2\left(v^2-u^2\right)\right)}dz_3, \tag{5.80}$$
$$A_3^1 = -\frac{2c\zeta_{22}c_2c_3\left(u^2\left(\zeta_{22}^2 - 2\zeta_{11}\right) - 2\zeta_{11}u\left(v^2-1\right) + v^2\left(2\zeta_{11} - \zeta_{22}^2\right)\right)}{\left(\zeta_{22}^2 - 2\zeta_{11}\right)\left(\zeta_{22}^2\left(u^2-v^2\right) - 2\zeta_{11}\left(u^2-1\right)\right)}dz_3, \tag{5.81}$$
$$A_3^2 = \frac{2c\zeta_{22}c_2s_3\left(-2\zeta_{11}u^2 + \zeta_{22}^2u^2 - 2\zeta_{11}u + 2\zeta_{11}uv^2 + 2\zeta_{11}v^2 - \zeta_{22}^2v^2\right)}{\left(\zeta_{22}^2 - 2\zeta_{11}\right)\left(2\zeta_{11} - 2\zeta_{11}u^2 + \zeta_{22}^2u^2 - \zeta_{22}^2v^2\right)}dz_3, \tag{5.82}$$
$$A_3^3 = \frac{2c\zeta_{22}s_2c_3\left(-2\zeta_{11}u^2 + \zeta_{22}^2u^2 - 2\zeta_{11}u + 2\zeta_{11}uv^2 + 2\zeta_{11}v^2 - \zeta_{22}^2v^2\right)}{\left(\zeta_{22}^2 - 2\zeta_{11}\right)\left(2\zeta_{11} - 2\zeta_{11}u^2 + \zeta_{22}^2u^2 - \zeta_{22}^2v^2\right)}dz_3. \tag{5.83}$$

## 5.4 Rod-structure analysis and the JMaRT solution

In this section, we will discuss the rod-structure of the above solution, which as we will see shortly, matches the fuzzball solution in [30]. Generally, the rod-structure of a $D$-dimensional solution with $D-2$ Killing vectors is determined by the specification of the rod intervals and the corresponding directions (see [76, 45, 50] and a short note in section 5.5.1). The directions are vectors $v_k$ for which

$$G(\rho = 0, z)v_k = 0, \qquad z \in \text{rod interval}, \tag{5.84}$$

where $G$ is the Killing part of the metric that we write using Weyl canonical coordinates $(\rho, z)$ as

$$ds^2 = f_E^2(d\rho^2 + dz^2) + G_{ij}(\rho, z)d\bar{x}^i d\bar{x}^j, \tag{5.85}$$

and $\bar{x}^i$ are the coordinates along the Killing directions.

We will use the rod diagram representations in [45] and "translate" the study of relation (5.84) to the $(u, v)$-coordinates as in [45]. In our case the $\{\bar{x}^i\}$ are

$\{\phi_-, y, \phi_+, t\}$ and thus the Killing part of the metric is a $4 \times 4$ matrix. The result, based on the calculations of the previous sections reads

$$G_{\text{Killing}} = \Big(\sqrt{\zeta_{11}\left(4c_2^2 - 2(u+1)\right) + \zeta_{22}^2(u+v)} \\ \sqrt{\zeta_{11}\left(4c_3^2 - 2(u+1)\right) + \zeta_{22}^2(u+v)}\Big)^{-1} g \tag{5.86}$$

with the $4 \times 4$ matrix $g$ having entries

$$g_{33} = \frac{1}{2\zeta_{11} - \zeta_{22}^2}\Big[2c^2\left(\zeta_{22}^2(u+v)\left(2s_2^2 + 2s_3^2 - u + v + 2\right)\right. \\ \left. +2\zeta_{11}\left(2s_2^2 - u + 1\right)\left(2s_3^2 - u + 1\right)\right)\Big], \tag{5.87a}$$

$$g_{34} = 2c\zeta_{22}s_2s_3(u+v), \tag{5.87b}$$

$$g_{35} = 2c\zeta_{11}v\left(2s_2^2 - u + 1\right)\left(-2s_3^2 + u - 1\right) + c\zeta_{22}^2(u+v)(uv-1), \tag{5.87c}$$

$$g_{36} = 2c\zeta_{22}c_2c_3(u+v), \tag{5.87d}$$

$$g_{44} = \zeta_{22}^2(u+v) - 2\zeta_{11}(u-1), \tag{5.87e}$$

$$g_{45} = -2\zeta_{11}\zeta_{22}s_2s_3(v+1), \tag{5.87f}$$

$$g_{46} = 0, \tag{5.87g}$$

$$g_{55} = \zeta_{11}\zeta_{22}^2\left(2u\left(s_2^2 + s_3^2 + 1\right) - 2v\left(s_2^2 + s_3^2 + 1\right) - 2u^2 + v^2 + 1\right) \\ - 2\zeta_{11}^2\left(2s_2^2 - u + 1\right)\left(-2s_3^2 + u - 1\right) + \frac{1}{2}\zeta_{22}^4(u-v)(u+v), \tag{5.87h}$$

$$g_{56} = 2\zeta_{11}\zeta_{22}c_2c_3(v-1), \tag{5.87i}$$

$$g_{66} = 2\zeta_{11}(u+1) - \zeta_{22}^2(u+v). \tag{5.87j}$$

The " mixed" angular coordinates that we have chosen to work with, namely $\phi_+ = \frac{1}{2}(\phi + \psi)$, $\phi_- = (\phi - \psi)$ give rise to non-standard orientations for the rod intervals $(-\infty, -c], [c, \infty)$, i.e. linear combinations of $\partial_{\bar{x}^i}$. In order to change to the standard Killing basis with $\phi, \psi$ as angular coordinates, we apply a linear coordinate transformation on $G_{\text{Killing}}$ as follows

$$G_{\text{final}} = \Lambda^T G_{\text{Killing}} \Lambda, \tag{5.88}$$

with $\Lambda$ a constant matrix in $\mathrm{SL}(4,\mathbb{R})$. We choose the matrix $\Lambda$ such that the desired asymptotic behaviour for the Killing metric becomes manifest. We have that

$$\Lambda = \begin{pmatrix} \frac{(\zeta_{22}^2 - 2\zeta_{11})}{4c} & 0 & -\frac{(\zeta_{22}^2 - 2\zeta 11)}{4c} & 0 \\ -\frac{1}{2}s_2s_3\zeta_{22} & 1 & \frac{1}{2}s_2s_3\zeta_{22} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2}c_2c_3\zeta_{22} & 0 & -\frac{1}{2}c_2c_3\zeta_{22} & 1 \end{pmatrix}, \text{ with action } \begin{pmatrix} z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \Lambda \begin{pmatrix} \phi \\ y \\ \psi \\ t \end{pmatrix} \tag{5.89}$$

is such a choice. Moreover, requiring the $z_5$ and $z_3$ coordinates to be asymptotically $\phi_+ = \frac{1}{2}(\phi + \psi)$ and $\phi_- = (\phi - \psi)$, we find that we must have

$$c = \frac{1}{4}\left(\zeta_{22}^2 - 2\zeta_{11}\right). \tag{5.90}$$

After the transformation (5.88) and the study of (5.84) we find that the rod-structure of the solution is:

- The semi-infinite rod $z \in (-\infty, -c]$ has orientation $\partial_\phi$.
- The middle rod $z \in [-c, c]$ has orientation $\partial_y + \frac{\zeta_{22}}{2\zeta_{11} s_2 s_3} \partial_\phi$.
- The semi-infinite rod $z \in [+c, \infty)$ has orientation $\partial_\psi$.

The above data correspond to the structure of the JMaRT solution. Indeed, as we will show now, the 2-charge solution that we computed can be identified with the singly rotating fuzzball in [30]. To see this, let us change to the coordinates used in [30], namely the polar coordinates $r, \theta$ and also identify the parameters $\zeta_{11}$ and $\zeta_{22}$ as

$$\zeta_{22} = -a_1, \quad \zeta_{11} = \frac{1}{2} M. \tag{5.91}$$

The prolate coordinates $(u, v)$ are related to $(r, \theta)$ as follows

$$u = \frac{2r^2}{a_1^2 - M} + 1, \qquad v = -\cos 2\theta, \tag{5.92}$$

or

$$r^2 = \frac{1}{2}(a_1^2 - M)(u - 1), \qquad \cos^2\theta = \frac{1}{2}(1 - v). \tag{5.93}$$

The final six-dimensional metric that we obtain is

$$\begin{aligned} ds_6^2 = & \frac{1}{\sqrt{\tilde{H}_2 \tilde{H}_3}} \left[ -(f - M)(dt - (f - M)^{-1} M c_2 c_3 a_1 \cos^2\theta d\psi)^2 + \right. \\ & \left. + f(dy + f^{-1} M s_2 s_3 a_1 \sin^2\theta d\phi)^2 \right] + \\ & + \sqrt{\tilde{H}_2 \tilde{H}_3} \left( \frac{dr^2}{r^2 + a_1^2 - M} + d\theta^2 + \frac{r^2 \sin^2\theta}{f} d\phi^2 + \frac{(r^2 + a_1^2 - M)\cos^2\theta}{f - M} d\psi^2 \right), \end{aligned} \tag{5.94}$$

with

$$f = r^2 + a_1^2 \sin^2\theta, \tag{5.95a}$$

$$\tilde{H}_2 = f + M \sinh^2\delta_2, \tag{5.95b}$$

$$\tilde{H}_3 = f + M \sinh^2\delta_3. \tag{5.95c}$$

We have used the same coordinates and parameters as [30]. The dilaton in six dimensions and the two-form are also found to match. The study of the smoothness properties of this configuration proceeds in exactly the same way as in [30].

#### Relation to the Myers-Perry instanton

We have mentioned in earlier sections that the JMaRT solution is obtained in our formalism through charging transformations of the over-rotating Myers-Perry instanton (i.e. the Euclidean version of the Myers-Perry solution with real poles). Let us look into this starting from (5.94). Taking the parameters $\delta_2, \delta_3$ to zero in (5.94), we obtain the over-rotating Myers-Perry metric lifted to six dimensions

$$\begin{aligned} ds_6^2 &= dy^2 - \left(1 - \frac{M}{f}\right)\left(dt - (f-M)^{-1} M a_1 \cos^2\theta d\psi\right)^2 + \\ &+ f\left(\frac{dr^2}{r^2 + a_1^2 - M} + d\theta^2\right) + r^2 \sin^2\theta d\phi^2 + \frac{f(r^2 + a_1^2 - M)\cos^2\theta}{f - M} d\psi^2. \end{aligned} \tag{5.96}$$

Now let us go back to (5.94) and shift the parameters $\delta_2, \delta_3$ as follows

$$\delta_2 = i\frac{\pi}{2} - \tilde{\delta}_2, \qquad \delta_3 = i\frac{\pi}{2} - \tilde{\delta}_3. \tag{5.97}$$

We get

$$\begin{aligned} ds_6^2 = &\frac{1}{\sqrt{\tilde{H}_2 \tilde{H}_3}}\left[-(f-M)(dt + (f-M)^{-1} M \tilde{s}_2 \tilde{s}_3 a_1 \cos^2\theta d\psi)^2 + \right. \\ &\left. + f(dy - f^{-1} M \tilde{c}_2 \tilde{c}_3 a_1 \sin^2\theta d\phi)^2\right] \\ &+ \sqrt{\tilde{H}_2 \tilde{H}_3}\left(\frac{dr^2}{r^2 + a_1^2 - M} + d\theta^2 + \frac{r^2 \sin^2\theta}{f} d\phi^2 + \frac{(r^2 + a_1^2 - M)\cos^2\theta}{f - M} d\psi^2\right), \end{aligned} \tag{5.98}$$

where

$$\tilde{H}_2 = f - M\cosh^2\tilde{\delta}_2, \tag{5.99a}$$

$$\tilde{H}_3 = f - M\cosh^2\tilde{\delta}_3. \tag{5.99b}$$

The shift $i\frac{\pi}{2}$ in (5.97) is related to the shift in the charging transformation (5.62). If we did not have it in the charging element, the solution we would have obtained would be (5.98). Now if we set the charge parameters $\tilde{\delta}_2, \tilde{\delta}_3$ to zero in (5.98) we get the metric

$$\begin{aligned} ds_6^2 = &-dt^2 + f(f-M)^{-1}(dy - f^{-1} M a_1 \sin^2\theta d\phi)^2 + \frac{(f-M) r^2 \sin^2\theta}{f} d\phi^2 \\ &+ (r^2 + a_1^2 - M)\cos^2\theta d\psi^2 + (f - M)\left(\frac{dr^2}{r^2 + a_1^2 - M} + d\theta^2\right). \end{aligned} \tag{5.100}$$

We recognize the above solution as the six-dimensional uplift of the Myers-Perry instanton if we make the replacement $r^2 \to \tilde{r}^2 = r^2 + a_1^2 - M$, i.e.

$$\begin{aligned} ds_6^2 = &-dt^2 + (\tilde{f} + M)\tilde{f}^{-1}(dy - (\tilde{f} + M)^{-1} M a_1 \sin^2\theta d\phi)^2 \\ &+ \frac{\tilde{f}(\tilde{r}^2 - a_1^2 + M)\sin^2\theta}{\tilde{f} + M} d\phi^2 + \tilde{r}^2 \cos^2\theta d\psi^2 + \tilde{f}\left(\frac{d\tilde{r}^2}{\tilde{r}^2 - a_1^2 + M} + d\theta^2\right). \end{aligned} \tag{5.101}$$

The Myers-Perry metric (5.96) is related to the Myers-Perry instanton (5.101) through analytic continuation, in particular through the following replacements in (5.96):

$$t \to iy, \quad y \to it, \quad a_1 \to -ia_1, \quad M \to -M, \quad \phi \leftrightarrow \psi, \quad \theta \to \frac{\pi}{2} - \theta, \quad r^2 \to \tilde{r}^2. \tag{5.102}$$

## 5.5 Addendum

### 5.5.1 On the rod-structure of solutions

The so-called rod-structure of solutions of the gravitational field equations arises from studying the behaviour of the Killing part of the metric (5.85) $G(\rho, z)$ at $\rho \to 0$ [76, 45, 50]. Since the condition on $G$ is (metric in canonical form)

$$\sqrt{|\det G|} = \rho, \tag{5.103}$$

we have that at least one eigenvalue of $G(0, z)$ must be zero, i.e.

$$\dim(\ker(G(0, z))) \geq 1. \tag{5.104}$$

However, if we assume that $G(0, z)$ has more than one zero eigenvalues, the curvature invariant is shown to diverge [45]. Therefore, to maintain a regular behaviour, we require that

$$\dim(\ker(G(0, z))) = 1, \tag{5.105}$$

everywhere except on isolated points on the $z$-axis. The values of $z$ where the kernel of $G(0, z)$ has dimension greater than one divide the $z$-axis into intervals that are called rods, or rod-intervals. Let us assume there are $n + 1$ such intervals $[c_{k-1}, c_k], k = 1, 2, ..., n + 1$ where $c_1, c_2, ..., c_n$ are isolated values of $z$ and $c_0 = -\infty, c_{n+1} = \infty$. For all rod-intervals $[c_{k-1}, c_k]$ we can find a vector

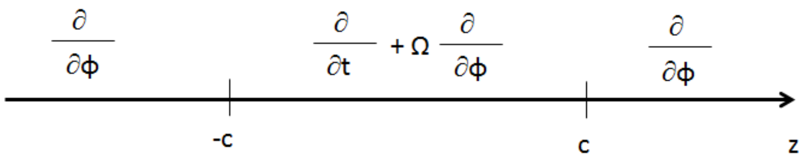
$$v_k = v_k^a \xi_{(a)} \tag{5.106}$$

such that

$$G(0, z)v_k = 0 \quad \text{for} \;\; z \in [c_{k-1}, c_k] \tag{5.107}$$

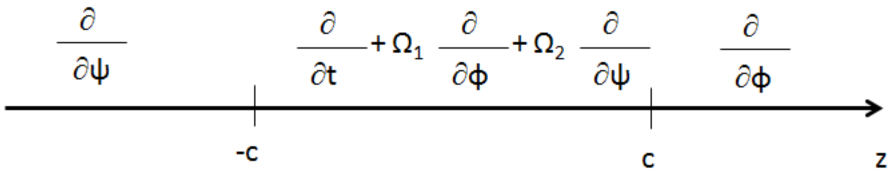
where $v_k$ is a vector in $\mathbb{R}^{D-2}$ for a solution characterized by $D - 2$ Killing vectors and $\xi_{(a)}$ is a basis of the $(D - 2)$-dimensional vector space spanned by the Killing vector fields. The vector $v_k$ is called the direction (or orientation) of the rod-interval $[c_{k-1}, c_k]$ and it is a non-zero vector (defined up to a multiplicative factor) for all $k \in \{1, 2, ..., n + 1\}$. The rod-intervals together with the respective directions $v_k$ constitute the *rod-structure* of a solution.

As an explanatory note let us give two examples of rod-structure diagrams. For the four-dimensional Kerr solution we can draw the following diagram where there are three rod-intervals and so two special values on the $z$-axis that we take to be $c_1 = -c\,, c_2 = c$ :



where on top of the rod-intervals are the rod-directions and $\Omega$ is the angular velocity of the Kerr black hole. As we can see from this diagram, there are two parameters characterizing the solution, i.e. $\Omega$, $c$ that are related to the mass and angular momentum of the black hole.

As an example in five dimensions, the Myers-Perry black hole corresponds to the following rod-diagram




where in this case there are two independent planes of rotation, corresponding to the $\phi$ and $\psi$ angles. The respective angular velocities are $\Omega_1, \Omega_2$. Together with $c$ these parameters fully characterize the Myers-Perry black hole.

The importance of the rod-structure analysis for solutions of pure $D$-dimensional gravity (as the ones above) lies in the conjecture that a solution is uniquely determined by the rod data. Among the general guidelines concerning the information that one extracts from a rod diagram are [50]

- finite rod intervals with timelike directions correspond to event horizons in spacetime while finite rod intervals with spacelike directions can be interpreted as Kaluza-Klein coordinates in the asymptotic region.
- semi-infinite rod intervals with spacelike directions $\partial_{x^i}$ indicate axes of rotational symmetry with $x^i$ the azimuthal angle.

In the case of the JMaRT solution, whose rod-structure was analysed in section 5.4, the finite rod has spacelike direction and indeed corresponds to the shrinking $y$-circle at $\rho = 0$ [30]. We note that the rod end points (isolated values of $z$) are identified with the pole positions in the soliton picture (this is true for all solitonic solutions). Moreover, comparing to the analysis in [30], we were able to relate the parameters in the BM vectors to the physical parameters in the (singly rotating) JMaRT solution. This shows that there is a connection of the BM vectors (similarly to the vectors $m_0^{(k)}$ in the Belinski-Zakharov method) to the rod orientations of the generated solution; however, the existence of an explicit relation is still not known.

### 5.5.2 A note on different $\mathrm{SL}(3, \mathbb{R})$ vacuum truncations

The vacuum gravity sector in the $D = 5$ Euclidean theory can be parameterized by the fields

$$U, \quad y^I = y, \quad \sigma, \quad \zeta^0, \quad \tilde{\zeta}_0. \tag{5.108}$$

while the rest vanish, i.e.

$$x^I = 0, \quad \zeta^I = 0, \quad \tilde{\zeta}_I = 0. \tag{5.109}$$

The metric uplifted to six dimensions reads

$$ds_6^2 = -dt^2 + ds_5^2 \tag{5.110}$$

and $ds_5^2$ is the Euclidean $D = 5$ metric. The scalar fields (5.108) form an $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(1,2)$ subspace of $\mathrm{SO}(4,4)/(\mathrm{SO}(2,2)\times\mathrm{SO}(2,2))$. The $\mathfrak{sl}(3,\mathbb{R})$ algebra is generated by

$$H_0, \quad H_1 + H_2 + H_3, \quad E_{q_0}, \quad E_{p^0}, \quad E_0 \tag{5.111}$$

and their transposes.

Now if we perform a reduction of the form[5]

$$ds_6^2 = dy^2 + d\tilde{s}_5^2, \tag{5.112}$$

such that $d\tilde{s}_5^2$ is a Lorentzian $D = 5$ metric, we find that the set of scalars parameterizing this type of truncation is

$$\tilde{\zeta}_0, \quad \tilde{\zeta}_1, \quad \chi^1, \quad y^1 = f^3 e^{-4U}, \quad y^2 = y^3 = e^{2U}. \tag{5.113}$$

The corresponding $\mathrm{SL}(3,\mathbb{R})$ group is now generated by the subset of $\mathrm{SO}(4,4)$ generators

$$H_1, \quad H_0 + H_2 + H_3, \quad F_{p^1}, \quad E_{p^0}, \quad E_1 \tag{5.114}$$

and their transposes.

The two different truncations discussed above, in terms of the associated $\mathrm{SL}(3,\mathbb{R})$ subgroups, are related by a conjugation relation in $\mathrm{SO}(4,4)$. This conjugation is by the element

$$w = e^{\frac{i\pi}{2}(E_{q_2}+E^\sharp_{q_2})+\frac{i\pi}{2}(E_{q_3}+E^\sharp_{q_3})} \tag{5.115}$$

and explains the shift by $\frac{i\pi}{2}$ in the charging parameter when we work with the Myers-Perry instanton rather than an over-rotating black hole.

### 5.5.3 Monodromy matrix for the Myers-Perry black hole

In light of the incorporation of five-dimensional asymptotically flat solutions to the BM method, it is useful to work out a standard example of a solitonic solution in five dimensions. To this end, we present for the first time the monodromy matrix for the (doubly rotating) Myers-Perry black hole [29], within the STU set-up.

[5] We note that the the $y$ coordinate in this discussion is the coordinate $z_6$ of chapter 4, but in this chapter it is the spatial direction over which we reduce the theory from $D = 4$ to $D = 3$ dimensions (cf. (5.16)).

We consider the standard form of the Myers-Perry metric given by

$$
\begin{aligned}
ds^2 =& - dt^2 + \frac{M}{\Sigma}\left(dt - a_1(1-x^2)d\phi - a_2 x^2 d\psi\right)^2 + (r^2+a_1^2)(1-x^2)d\phi^2 \\
&+ (r^2+a_2^2)x^2 d\psi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\frac{dx^2}{(1-x^2)}
\end{aligned} \tag{5.116}
$$

where $x = \cos\theta$ and

$$
\Delta = r^2\left(1+\frac{a_1^2}{r^2}\right)\left(1+\frac{a_2^2}{r^2}\right) - M, \tag{5.117}
$$

$$
\Sigma = r^2 + a_1^2 x^2 + a_2^2(1-x^2) \tag{5.118}
$$

with $M$ the mass parameter and $a_1, a_2$ the rotational parameters corresponding to the two independent planes of rotation of the Myers-Perry black hole. Next, we change to the angular coordinates $\phi_+, \phi_-$ (5.31) in order to avoid divergent asymptotic behaviour of the matrix M (cf. section 5.2). With this change of coordinates the metric (5.116) becomes

$$
\begin{aligned}
ds^2 =& - dt^2 + \frac{\Sigma}{\Delta}dr^2 + \frac{\Sigma}{1-x^2}dx^2 + \frac{1}{4}(a_1^2+r^2)(1-x^2)(d\phi_- - 2d\phi_+)^2 \\
&+ \frac{1}{2}(a_2^2+r^2)x^2(d\phi_- + 2d\phi_+)^2 + \frac{1}{\Sigma}\Big[M\Big(dt - \frac{1}{2}(a_2x^2 - a_1(1-x^2))d\phi_- \\
&- (a_1(1-x^2) + a_2x^2)d\phi_+\Big)^2\Big].
\end{aligned} \tag{5.119}
$$

The monodromy matrix associated to the above solution is a meromorphic function with two poles, in accordance with the rod diagram in section 5.5.1. Moreover, it can be written in the form (5.28a) with residue matrices of rank two and fulfills the requirement of five-dimensional asymptotically flat behaviour as discussed in section 5.2. The entries of this monodromy matrix $\mathcal{M}(w)$ are found to be

$$
\mathcal{M}_{11} = \frac{1}{w^2-c^2}\left(\frac{1}{8}(a_2^2 - a_1^2 - M - 4w)\right) \tag{5.120}
$$

$$
\mathcal{M}_{14} = \frac{1}{w^2-c^2}\left(\frac{-a_1M}{4}\right) \tag{5.121}
$$

$$
\mathcal{M}_{17} = \frac{1}{w^2-c^2}\left(\frac{1}{16}\left(a_1^4 + 4a_1a_2M + a_1^2(-2a_2^2+M) + (a_2^2 - M - 4w)(a_2^2+4w)\right)\right) \tag{5.122}
$$

$$
\mathcal{M}_{33} = \frac{1}{w^2-c^2}\left(\frac{1}{8}\left(a_2^2 - a_1^2 + M - 4w\right)\right) \tag{5.123}
$$

$$
\mathcal{M}_{35} = \frac{1}{w^2-c^2}\left(\frac{1}{16}\left(-a_1^4 - 4a_1a_2M + a_1^2(2a_2^2+M) - (a_2^2 + M - 4w)(a_2^2+4w)\right)\right) \tag{5.124}
$$

$$
\mathcal{M}_{38} = \frac{1}{w^2-c^2}\left(\frac{a_2M}{4}\right) \tag{5.125}
$$

$$\mathcal{M}_{41} = -\mathcal{M}_{14} \tag{5.126}$$

$$\mathcal{M}_{44} = \frac{1}{w^2 - c^2}\left(\frac{1}{16}\left((-a1^4 - a2^4 + 2a1^2(a2^2 + 2M) + (M - 4w)^2\right)\right) \tag{5.127}$$

$$\mathcal{M}_{47} = \frac{1}{w^2 - c^2}\left(\frac{1}{8}\left(-M(a_1^3 + a_1^2 a_2 + a_2(-a_2^2 + M - 4w) - a_1(a_2^2 + 4w))\right)\right) \tag{5.128}$$

$$\mathcal{M}_{53} = -\mathcal{M}_{35} \tag{5.129}$$

$$\begin{aligned}\mathcal{M}_{55} = \frac{1}{w^2 - c^2}\Big(\frac{1}{32}\Big(&M(4a_1^4 + 8a_1^3 a_2 - 4a_2^4 - M(M - 4w) - a_2^2(-3M + 16w)\\ &- a_1^2(3M + 16w) - 8a_1(a_2^3 + 4a_2 w))\Big)\Big)\end{aligned} \tag{5.130}$$

$$\mathcal{M}_{58} = \frac{1}{w^2 - c^2}\left(\frac{1}{8}\left(M(-a_1^3 - a_1^2 a_2 + a_2^3 + 4a_2 w + a_1(a_2^2 + M + 4w))\right)\right) \tag{5.131}$$

$$\mathcal{M}_{71} = -\mathcal{M}_{17} \tag{5.132}$$

$$\mathcal{M}_{74} = \mathcal{M}_{47} \tag{5.133}$$

$$\begin{aligned}\mathcal{M}_{77} = \frac{1}{w^2 - c^2}\Big(\frac{1}{32}\Big(&M(4a_1^4 + 8a_1^3 a_2 - 4a_2^4 + M(M + 4w) - a_2^2(-3M + 16w)\\ &- a_1^2(3M + 16w) - 8a_1(a_2^3 + 4a_2 w)\Big)\Big)\end{aligned} \tag{5.134}$$

$$\mathcal{M}_{83} = \mathcal{M}_{38} \tag{5.135}$$

$$\mathcal{M}_{85} = -\mathcal{M}_{58} \tag{5.136}$$

$$\mathcal{M}_{88} = \frac{1}{w^2 - c^2}\left(\frac{1}{16}\left(-(a_1^2 - a_2^2)^2 + 4a_2^2 M + (M + 4w)^2\right)\right), \tag{5.137}$$

while $\mathcal{M}_{22} = \mathcal{M}_{66} = 1$ and all other entries are zero. The pole position $c$ is related to $M, a_1, a_2$ as

$$c = \frac{1}{4}\sqrt{a_1^4 + (a_2^2 - M)^2 - 2a_1^2(a_2^2 + M)}. \tag{5.138}$$

The monodromy matrix can be brought to the form (5.28a) with rank-2 residue matrices parameterized as in (5.48). The vectors $\boldsymbol{a_1}, \boldsymbol{a_2}$ are given by

$$\boldsymbol{a_1} = \left(1, 0, 0, -\frac{a_1^2 - a_2^2 - 4c + M}{2a_1}, 0, 0, \frac{(a_1 + a_2)(a_1^2 - a_2^2 - 4c) + a_2 M}{2a_1}, 0\right) \tag{5.139}$$

$$\boldsymbol{a_2} = \left(1, 0, 0, -\frac{a_1^2 - a_2^2 + 4c + M}{2a_1}, 0, 0, \frac{(a_1 + a_2)(a_1^2 - a_2^2 + 4c) + a_2 M}{2a_1}, 0\right). \tag{5.140}$$

We note that the limit $a_1 \to 0, a_2 \to 0$ should be taken at the level of the monodromy matrix, whereby for $a_1 = a_2 = 0$ we obtain the monodromy for the Schwarzschild black hole in five dimensions (in the coordinates (5.31)). From this limit as a starting point, we find suitable $\boldsymbol{a_k}$ vectors and not from setting the rotational parameters to zero in (5.139),(5.140) (which would lead to divergent expressions).

The vectors $\boldsymbol{b_1}, \boldsymbol{b_2}$ are constructed as described in section 5.3, (cf. (5.50),(5.51)) and read

$$\begin{aligned}\boldsymbol{b_1} = \Big(&\frac{c}{a_1^2}\big(2a_1^4 + 4a_1^3 a_2 - (2a_2^2 - M)(a_2^2 + 4c - M) - 2a_1 a_2(2a_2^2 + 8c - M)\\ &- a_1^2(8c + M)\big), 0, 0, \frac{2c(a_1^3 + a_1^2 a_2 - a_2(a_2^2 - 4c + M) - a_1(a_2^2 + 4c + M))}{a_1^2},\\ &0, 0, \frac{-2c((a_1 + a_2)^2 + 4c - M))}{a_1^2}, 0\Big) \qquad (5.141)\end{aligned}$$

$$\begin{aligned}\boldsymbol{b_2} = \Big(&\frac{c}{a_1^2}\big(2a_1^4 + 4a_1^3 a_2 - (2a_2^2 - M)(a_2^2 - 4c - M) + 2a_1 a_2(-2a_2^2 + 8c + M)\\ &+ a_1^2(8c - M)\big), 0, 0, \frac{2c(a_1^3 + a_1^2 a_2 - a_1(a_2^2 - 4c + M) - a_2(a_2^2 - 4c - M))}{a_1^2},\\ &0, 0, \frac{-2c((a_1 + a_2)^2 - 4c - M)}{a_1^2}, 0\Big). \qquad (5.142)\end{aligned}$$

The constant parameters $\alpha_k, \beta_k$ are given by (5.52) and the rest of the factorization proceeds as in section 4.2. We note that, the $\gamma_k$ and the matrix $\Gamma$ remain unchanged under constant transformations of the monodromy matrix of the type (5.44). The final result is the matrix $M(x)$ given by (5.29), from which we reconstruct the Myers-Perry solution in the form (5.119).

# Chapter 6

# Conclusions

In this thesis, we have studied gravity-matter systems that are completely integrable in two dimensions. These systems have matter described by a non-linear $\sigma$-model with target manifold $\mathrm{G}_E/\mathrm{K}_E$. This structure emerges already in three dimensions, where the symmetry associated to the Ehlers group $\mathrm{G}_E$ is a finite, global symmetry of the Lagrangian. In two dimensions, at the level of the equations of motion, the symmetry becomes infinite dimensional and is connected to the affine extension of the group $\mathrm{G}_E$; this is the Geroch symmetry. The complete integrability of the two-dimensional equations of motion is exhibited by the existence of a linear system (Lax pair), first formulated by Belinski and Zakharov for Einstein gravity and later by Breitenlohner and Maison, who chose an equivalent but strictly group theoretic approach. Both linear systems can be solved (in the soliton sector) through the inverse scattering method that is applied in a different way in each approach, but ultimately solves the same problem. The BZ inverse scattering method uses a dressing technique to generate solutions and is the most practical and easily applicable method for solution generation in four- and five- dimensional vacuum gravity. However, if one wants to treat extended gravity theories in the $\mathrm{G}_E/\mathrm{K}_E$ class, such as certain supergravity models descending from string theory, the BZ method cannot be systematically applied. The reason originates in the fact that, as a result of the BZ dressing transformation, the generated solutions do not satisfy the coset constraints and need additional scaling factors to turn them into "physical" solutions. In $D > 4$, as was explained in chapter 3, below (3.44), the fractional powers of $\rho$ that enter the expression of the coset metric typically lead to singular behaviour of the final solution. In pure $D = 5$ gravity, Pomeransky [46] found a way around this problem but beyond this case, it would be useful to have a method that evades this problem. Inspired by the possibility to formulate a systematic solution generating technique that can be generalized and applied to a broader set of gravity models, we focused on the BM approach in this work. Starting from Einstein gravity, we then set out to extend it to STU supergravity, where with appropriate technical adjustments we were able to reconstruct the four-dimensional Cvetic-Youm solution and the JMaRT fuzzball in six dimensions.

After an introductory basis on the symmetries of dimensionally reduced gravity, we studied the linear system of Einstein gravity reduced to two dimensions from the Belinski-Zakharov point of view as well as that of Breitenlohner-Maison. With the interrelations of the linear systems in the Ehlers coset already established in [47], we extended this by studying the relation of the respective generating functions, both in the Ehlers and Matzner-Misner coset (see (3.31),(3.32)). In particular for the relation (3.32), it should be noted that the factor appearing there is modified with respect to [13], such that it maps the generating functions away from $t \to 0$ as well. However, there are certain comments in order concerning these relations. The BZ generating function that we relate to the BM one, should be the "physical" one, i.e. the one satisfying the right determinant constraint at the limit $\lambda \to 0$. This is the direct product of the BZ method however; due to the dressing procedure that is not restricted to ensure that the solutions are physical, there is the need to rescale the solution at the end. What is more, both of the generating functions that solve the respective linear systems are in fact members of an equivalence class of solutions up to a gauge. This makes it harder to match individual solutions to one another. Therefore, in the matter of relating the BZ and BM methods, we arrived at the conclusion that, the link between them is not useful for practical purposes; we can only speak of representative relations and not directly view the BZ approach as an implementation of the Geroch group.

Turning to the study of the BM approach that keeps the group theoretic structure manifest, we implement the solution generating technique described in their unpublished notes [26]. For solitonic solutions, given a seed solution, BM show that through a set of purely algebraic steps a new solution can be generated. They have developed this method for the case of Einstein gravity, that is connected to the $\mathrm{SL}(2,\mathbb{R})$ Ehlers group as well as for the more general case of $\mathrm{SL}(n,\mathbb{R})$. In the class of stationary, axisymmetric solutions that we focus on in this work, black holes are among the two-soliton solutions obtained through this technique. The Schwarzschild and Kerr solutions were worked out within this framework by Breitenlohner and Maison [26],[27]. As a next step, we constructed the Kerr-NUT solution starting from the most general two-soliton ansatz and including the parameters needed to account for the mass, angular momentum and NUT charge characterizing this black hole. Apart from a novel example constructed with the BM method, this exercise served as a basis for the understanding of further applications involving groups other than $\mathrm{SL}(n,\mathbb{R})$.

In the class of $\mathrm{G}_E/\mathrm{K}_E$ models that become completely integrable in two dimensions is the STU model, an $\mathcal{N}=2$ supergravity theory in four dimensions. In the bosonic sector, the model admits interesting (non-extremal) black hole solutions, among which the Cvetic-Youm four-charge black hole. The task in this case was to adjust the BM method to the group structure associated to the symmetries of the dimensionally reduced STU theory. Through the study of solutions in the pure gravity sector (no electromagnetic charges) such as the Kerr solution, we observed that the corresponding $\mathrm{SO}(4,4)$ monodromy matrix contains the $\mathrm{SL}(2,\mathbb{R})$ Kerr monodromy twice. This structure gave rise to rank-two residues in the solitonic expansion (4.80) and this remained so for the four-charge solution. We are inclined to believe that this is the case in general for solutions of physical interest in this context. Taking the increased rank into account as well as the group properties of the relevant coset space $(\mathrm{SO}(4,4)/(\mathrm{SO}(2,2) \times \mathrm{SO}(2,2)))$, we modified the BM method accordingly. Making guesses educated by the $\mathrm{SL}(2,\mathbb{R})$ case, we chose suitable vectors involved in the parameterization of the residue matrices and constructed the Kerr solution within the STU model. Next, in order to add charges to this solution, we applied constant charging transformations in the subgroup $\mathrm{K}_E$ that had the effect of rotating the vectors but leaving the $\Gamma$ matrix unchanged. With the right manipulations, involving coordinate transformations, field dualizations and parameter redefinitions, we could recognize the Cvetic-Youm four-charge solution. Based on the observation of the residue-rank being higher in the $\mathrm{SO}(4,4)$ case, we extended the BM algorithm to account for residue matrices with rank $r \geq 1$. This generalisation might be useful for other cases involving different groups.

The next challenge that we chose to pursue was the treatment of solutions with flat asymptotics in five dimensions, within STU supergravity. This requires a non-trivial modification in the BM method. Specifically, the solitonic ansatz for the monodromy matrix must have appropriate asymptotic behaviour that is different from the one in the four dimensional case. Following the discussion on the $\mathrm{SL}(3,\mathbb{R})$ case connected to five-dimensional vacuum gravity in [63], we were able to extend this to $\mathrm{SO}(4,4)$ and thus incorporate five-dimensional asymptotics in the BM method (using a suitable coordinate system that avoids poles at infinity in the soliton ansatz [63]). In turn, this development allowed for the inverse scattering construction of the Myers-Perry solution and its charged versions obtained through suitable charging transformations, similarly to our previous STU constructions. The main endeavour however, was to achieve the construction of so-called fuzzball solutions through the six-dimensional uplifting of STU supergravity. It turned out that if we require the poles of the monodromy matrix to be real, the fuzzball solution cannot be obtained as an uplift of the charged under-rotating Myers-Perry solution. Instead, we found that in our construction, the (singly rotating) JMaRT solution in [30] is obtained from charging up the Euclidean Myers-Perry instanton in the parameterization (5.97) and trivially lifting it to six dimensions. For this reason, we have chosen a series of dimensional reductions such that the first reduction from six to five dimensions is over a timelike Killing vector; this leads to Euclidean five-dimensional STU supergravity. Within this theory, we construct the Myers-Perry instanton using the BM soliton algorithm. This was the key part in our two-charge fuzzball calculation. The addition of charges was done by the action of a constant element in a group conjugate to the denominator subgroup $\mathrm{K}_E$, determined by the requirement that the transformation preserve the asymptotic behaviour of five-dimensional asymptotically flat solutions. Moreover, once we reached the solution and changed to standard angular coordinates, we performed the rod-structure analysis of the JMaRT singly rotating fuzzball, in agreement with [30] (cf. relations (3.7) and (3.9) in [30]).

We have mentioned that part of our motivation for this work was to develop a systematic solution generating technique for gravity theories where such a technique is not available. Indeed the (group theoretic) nature of the BM method is expected and shown to be extendible and of broad applicability. However, certain drawbacks have to be noted. The biggest one is the matter of choosing suitable vectors in the soliton ansatz which correspond to solutions of physical interest. Determining the vectors (or their general form) for known solutions is fairly straightforward, since the monodromy matrix can be deduced through the behaviour of the coset metric $M$ on the $z$-axis (see section 4 in [13] and [75]). Beyond known solutions, it becomes much trickier to guess suitable vectors. Of course, in principle one can generate the most general soliton solution with vectors written in a completely general form, but the final metric would be something very hard to analyse and recognize.

Another issue concerning the BM set-up that we focused on is that it is only adequately developed in the Ehlers coset. On one hand, this simplifies things in the algorithm itself and allows for a very simple form of seed solution (flat space as a seed is represented by the unit matrix for solutions with flat asymptotics in four dimensions) ; on the other hand, it makes computations more involved at the end, when fields that appear in the final spacetime metric have to be extracted through integrations of duality relations. This becomes especially tricky when solutions grow more and more complicated (multi-charged configurations in supergravity). In principle, the BM method can also be formulated in the Matzner-Misner coset and thus provide immediate information for the final solution. However, this would in turn introduce complications in the Riemann-Hilbert factorization process that mainly reside on the issue of including poles at infinity (for Einstein gravity and the Matzner-Misner $\mathrm{SL}(2,\mathbb{R})$, the coset metric has explicit powers of $\rho$ that lead to infinities in the flat space monodromy matrix, see (2.7) in [13] and section 4).

The latter issue on including poles at infinity arose also in our work described in chapters 4, 5 and was related to the choice of angular coordinates in the Killing part of the metric. It turned out that using standard angular coordinates $\phi, \psi$ introduced infinities in the asymptotic behaviour of the coset metric. Perhaps if the technical modification for treating poles at infinity in the soliton ansatz is made, computations become simpler; this would also allow for non-trivial seed solutions.

All the computations summarized above and given in more detail throughout this thesis count as tests of the capacity of the BM method to generate physically interesting solutions. Since five-dimensional asymptotics can be incorporated in the BM formalism, one natural direction to test the method further is to construct other black hole objects in five dimensions, such as black rings [77, 78] and "black Saturns"[79]. Another direction worth exploring from the point of view of BM inverse scattering is the case of extremal solutions. It could be interesting to understand extremality through inverse scattering data. The most desirable direction of course would be to generate new solutions that are physically meaningful. As mentioned earlier, this depends a lot on choosing the right vectors and there is yet not a clear recipe to do so. To make progress in this direction, one could try taking smaller steps towards generating new solutions. For example, generalizations of our JMaRT inverse scattering construction, from the addition of a second rotational parameter to multi-center non-supersymmetric fuzzballs. This direction could be especially promising, since the fuzzball proposal aspires to explain the nature of black hole entropy and provide a way out of the information paradox [65]. We believe that, by extending the study on the formalism itself and the generation of more solutions, this work can carry on further and become a promising direction in the field of solution generation.

# Bibliography


[1] J. Ehlers. Transformations of Static Exterior Solutions of Einstein's Gravitational Field Equations into Different Solutions by Means of Conformal Mappings. *Les théories physiques de la gravitation.* CNRS, Paris, 1959.

[2] R.P. Geroch. A Method for generating solutions of Einstein's equations. *J. Math. Phys.*, 12:918–924, 1971.

[3] R.P. Geroch. A Method for generating new solutions of Einstein's equation. 2. *J. Math. Phys.*, 13:394–404, 1972.

[4] C. Hoenselaers, W. Kinnersley, and B. Xanthopoulos. *J. Math. Phys.* **20** (1979) 2530.

[5] B. K. Harisson. Bäcklund Transformations for the Ernst Equations of General Relativity. *Phys. Rev. Lett.* **41** (1978) 1197.

[6] I. Hauser and F.J. Ernst. Proof of a Geroch conjecture. *J. Math. Phys.* **22** (1981) 1051.

[7] C. M. Cosgrove. Relationships Between The Group Theoretic And Soliton Theoretic Techniques For Generating Stationary Axisymmetric Gravitational Solutions. *J. Math. Phys.* **21** 2417 (1980).

[8] C. M. Cosgrove. Relationships between the inverse scattering techniques of Belinski-Zakharov and Hauser-Ernst in general relativity. *J. Math. Phys.* **23** 615 (1982).

[9] V.A. Belinsky and V.E. Zakharov. Integration of the Einstein Equations by the Inverse Scattering Problem Technique and the Calculation of the Exact Soliton Solutions. Sov. Phys. JETP **48** (1978) 985, Zh. Eksp. Teor. Fiz. **75** (1978) 1953.

[10] V.A. Belinsky and V.E. Sakharov. Stationary Gravitational Solitons with Axial Symmetry. Sov. Phys. JETP **50** (1979) 1-9, Zh. Eksp. Teor. Fiz. **77** (1979) 3-19.

[11] V. Belinski and E. Verdaguer. *Gravitational Solitons.* Cambridge University Press, Cambridge, 2001.

[12] D. Kramer, H. Stephani, E. Herlt, M. MacCallum, and E. Schmutzer. *Exact Solutions of Einstein's Field Equations.* Cambridge University Press, Cambridge, 1980.

[13] P. Breitenlohner and D. Maison. On the Geroch Group. *Ann. Inst. Henri Poincaré*, 46:215–246, 1987.

[14] P. Breitenlohner, D. Maison, and G.W. Gibbons. Four-Dimensional black holes from Kaluza-Klein theories. *Commun. Math. Phys*, 120:295–333, 1988.

[15] B. Julia. Infinite Lie Algebras In Physics. In **Baltimore 1981, Proceedings, Unified Field Theories and Beyond**, pages 23–41. Conf. Proc. C **8006162** (1980) 331.

[16] B. Julia. Gravity, Supergravities and Integrable Systems. In **Istanbul 1982, Proceedings, Group Theoretical Methods In Physics**, pages 214–227. Paris Ec. Norm. Sup. - LPTENS 82-23 (82,REC.MAR.83).

[17] W. Kinnersley and D. M. Chitre, Symmetries of the Stationary Einstein-Maxwell Field Equations. 1. *J. Math. Phys.* **18** (1977) 1529. W. Kinnersley and D. M. Chitre, Symmetries of the Stationary Einstein-Maxwell Field Equations. 2. *J. Math. Phys.* **18** (1977) 1538. W. Kinnersley and D. M. Chitre, Symmetries of the Stationary Einstein-Maxwell Field Equations. 3. *J. Math. Phys.* **19** (1978) 1926.

[18] C. Devchand and D. B. Fairlie, A generating Function For Hidden Symmetries of Chiral Models, *Nucl. Phys. B* **194** (1982) 232. C. Devchand and J. Schiff, Hidden symmetries of the principal chiral model unveiled *Commun. Math. Phys.* **190** (1998) 675 [hep-th/9611081].

[19] H. Nicolai. The Integrability of N=16 Supergravity. *Phys. Lett. B.* **194** 402 (1987).

[20] J. H. Schwarz. Classical symmetries of some two-dimensional models coupled to gravity. *Nucl. Phys. B.* **454** (1995) 427 [hep-th/9506076].

[21] H. Nicolai, D. Korotkin, and H. Samtleben. Integrable classical and quantum gravity. In *Cargese 1996, Quantum fields and quantum space time* 203-243, [hep-th/9612065].

[22] D. Bernard and B. Julia. Twisted self-duality of dimensionally reduced gravity and vertex operators. *Nucl. Phys. B.* **547** (1999) 427 [hep-th/9712254].

[23] H. Nicolai and H. Samtleben. Integrability and canonical structure of d = 2, N=16 supergravity. *Nucl.Phys. B*, 533:210–242, 1998. [hep-th/9804152].

[24] H. Lu, M. J. Perry, and C. N. Pope. Infinite-dimensional symmetries of two-dimensional coset models coupled to gravity. *Nucl. Phys. B.* **806** (2009) 656 [arXiv:0712.0615 [hep-th]].

[25] V. G. Kac. *Infinite Dimensional Lie Algebras.* Cambridge University Press, Cambridge, 1990.

[26] P. Breitenlohner and D. Maison. Solitons in Kaluza-Klein Theories. Unpublished notes, June 1986.

[27] D. Maison. Duality and Hidden Symmetries in Gravitational Theories. *Lecture Notes in Physics*, 540:273–323, 2000.

[28] M. Cvetic and D. Youm. Entropy of nonextreme charged rotating black holes in string theory. *Phys. Rev. D.* **54** 2612 (1996) [hep-th/9603147].

[29] R.C. Myers and M.J. Perry. Black Holes in Higher Dimensional Space-Times. *Ann. Phys. (N.Y.)*, 172:304–347, 1986.

[30] V. Jejjala, O. Madden, S. F. Ross, and G. Titchener. Non-supersymmetric smooth geometries and D1-D5-P bound states. *Phys. Rev. D.* **71** 124030 (2005) [hep-th/0504181].

[31] H. Nicolai. Two-dimensional Gravities and Supergravities as Integrable Systems. In *Recent Aspects of Quantum Fields: Proceedings of the XXX Int. Universitätswochen für Kernphysik, Schladming, Austria*, pages 231–273, February and March 1991.

[32] H. Nicolai and A. Nagar. Infinite-dimensional symmetries in gravity. In I. Ciufolini, V. Gorini, U. Moschella, and F. Pietro, editors, *Gravitational Waves.* Institute of Physics Publishing, Bristol, Philadelphia, 2001.

[33] R. A. Matzner and C. W. Misner. Gravitational Field Equations for Sources with Axial Symmetry and Angular Momentum. *Phys. Rev.*, 154:1229–1232, 1967.

[34] R.M. Wald. *General Relativity.* The University of Chicago Press, Chicago, London, 1984.

[35] K. Iwasawa. On some types of topological groups. *Annals of Mathematics* (2) **50** (1949) 507558.

[36] A. Knapp. Lie Groups Beyond an Introduction. Progress in Mathematics Vol.140, ISBN 978-0-8176-4259-4.

[37] C. N. Pope. Kaluza-Klein theory. http://people.physics.tamu.edu/pope/ihplec.pdf [accessed on 9/9/2014].

[38] H. Nicolai and O. Schlotterer. Lecture notes on *Infinite Dimensional Symmetries*, 15th Saalburg Summer School, Wolfersdorf, Thuringia, September 2009.

[39] B. Zumino. Anomalies, Cocycles and Schwinger terms. In Proceedings of the Symposium on *Anomalies, Geometry and Topology*, Argonne-Chicago, 1985.

[40] B. Y. Hou and W. Li. Virasoro algebra in the solution space of the Ernst equation. *Lett. Math. Phys.* **13** (1987) 1.

[41] D. Maison. Geroch Group and Inverse Scattering Method. MPI-PAE/Pth-80/88.

[42] B. Julia and H. Nicolai. Conformal internal symmetry of 2-d sigma models coupled to gravity and a dilaton. *Nucl. Phys. B.* **482** (1996) 431 [hep-th/9608082].

[43] S. Nomikov, S.V. Manakov, L.P. Pitaevskii, and V.E. Zakharov. *Theory of Solitons: The Inverse Scattering Method.* Consultants Bureau, New York and London, 1984.

[44] D. Katsimpouri, A. Kleinschmidt, and A. Virmani. Inverse Scattering and the Geroch Group. *JHEP.* **1302** (2013) 011 [arXiv: 1211.3044 [hep-th]].

[45] T. Harmark. Stationary and Axisymmetric Solutions of Higher-Dimensional General Relativity. *Phys.Rev.D.* **70** 124002 (2004) [arXiv:hep-th/0408141].

[46] A. A. Pomeransky. Complete integrability of higher-dimensional Einstein equations with additional symmetry and rotating black holes. *Phys. Rev. D.* **73** 044004 (2006) [hep-th/0507250].

[47] P.Figueras, E. Jamsin, J. V. Rocha, and A. Virmani. Integrability of Five Dimensional Minimal Supergravity and Charged Rotating Black Holes. *Class. Quant. Grav.* **27** (2010) 135011 [arXiv: 0912.3199 [hep-th]].

[48] J.V. Rocha, M.J. Rodriguez, O. Varela, and A. Virmani. Inverse Scattering Construction of Dipole Black Rings. Springer Proc.Math.Stat. 60 (2014) 107-115.

[49] S. Beheshti and A. S. Tahvildar-Zadeh. Integrability and Vesture for Harmonic Maps into Symmetric Spaces. [arXiv: 1209.1383 [math.AP]].

[50] R. Emparan and H. S. Reall. Black Holes in Higher Dimensions. *Living Rev. Rel.* **11** 6 (2008) [arXiv: 0801.3471[hep-th]].

[51] D. Katsimpouri, A. Kleinschmidt, and A. Virmani. An Inverse Scattering Formalism for STU Supergravity. *JHEP.* **1403** (2014) 101 [arXiv: 1311.7018 [hep-th]].

[52] D.D.K. Chow and G. Compére. Seed for general rotating non-extremal black holes of $N$=8 supergravity. *Class. Quant. Grav.* **31** (2014) 022001 [arXiv:1310.1925].

[53] D.D.K. Chow and G. Compére. Black holes in N=8 supergravity from $SO(4,4)$ hidden symmetries. *Phys. Rev. D.* **90** 025029 (2014) [arXiv:1404.2602].

[54] M. Cvetic and D. Youm. Dyonic BPS Saturated Black Holes of Heterotic String on a Six-Torus. *Phys.Rev.D*, 53:584–588, 1996. [arXiv:hep-th/9507090].

[55] G.T. Horowitz, D.A. Lowe, and J.M. Maldacena. Statistical Entropy of Nonextremal Four-Dimensional Black Holes and U-Duality. *Phys.Rev.Lett.*, 77:430–433, 1996. [arXiv:hep-th/9603195].

[56] K. Behrndt, R. Kallosh, J. Rahmfeld, M. Shmakova, and W. K. Wong. STU black holes and string triality. *Phys. Rev. D.* **54** 6293 (1996),[hep-th/9608059].

[57] Z.-W. Chong, M. Cvetic, H. Lu, and C. N. Pope. Charged rotating black holes in four-dimensional gauged and ungauged supergravities. *Nucl. Phys. B.* **717** 246 (2005) [hep-th/0411045].

[58] A. Virmani. Subtracted Geometry From Harrison Transformations. *JHEP.* **1207** (2012) 086 [arXiv: 1203.5088 [hep-th]].

[59] M. J. Duff and R. R. Khuri. Four-dimensional string/string duality. *Nucl. Phys. B.* **411** 473 (1994) [hep-th/9305142].

[60] M.J. Duff, J.T. Liu, and J.Rahmfeld. Four-dimensional string-string-string triality. *Nucl. Phys. B.* **459** 125 (1996) [hep-th/9508094].

[61] G. Bossard, Y. Michel, and B. Pioline. Extremal black holes, nilpotent orbits and the true fake superpotential. *JHEP.* **1001** (2010) 038, [arXiv:0908.1742[hep-th]].

[62] A. Ceresole, R. D'Austria, S. Ferrara, and A. Van Proeyen. Duality transformations in supersymmetric Yang-Mills theories coupled to supergravity. *Nucl. Phys. B.* **444** 92 (1995) [hep-th/0701176].

[63] S. Giusto and A. Saxena. Stationary axisymmetric solutions of five dimensional gravity. *Class. Quant. Grav.* **24** 4269-4294, 2007, [arXiv:0705.4484 [hep-th]].

[64] D. Katsimpouri, A. Kleinschmidt, and A. Virmani. An Inverse Scattering Construction of the JMaRT Fuzzball. *JHEP.* **1412** (2014) 070 [arXiv: 1409.6471 [hep-th]].

[65] S. D. Mathur. The Fuzzball proposal for black holes: An elementary review. *Fortsch. Phys.* **53** 793 (2005) [hep-th/0502050].

[66] I. Bena, S. Giusto, C. Ruef, and N. P. Warner. A (Running) Bolt for New Reasons. *JHEP.* **0911** (2009) 089 [arXiv:0909.2559 [hep-th]].

[67] N. Bobev and C. Ruef. The Nuts and Bolts of Einstein-Maxwell Solutions. *JHEP.* **1001** (2010) 124 [arXiv:0912.0010 [hep-th]].

[68] S. Giusto, S. F. Ross, and A. Saxena. Non-supersymmetric microstates of the D1-D5-KK system. *JHEP.* **0712** 065 (2007) [arXiv:0708.3845 [hep-th]].

[69] S. Banerjee, B. D. Chowdhury, B. Vercnocke, and A. Virmani. Non-supersymmetric Microstates of the MSW System. *JHEP*. **1405** 011 (2014) [arXiv:1402.4212 [hep-th]].

[70] A. Sahay and A. Virmani. Subtracted Geometry from Harrison Transformations: II. *JHEP*. **1307** (2013) 089 [arXiv:1305.2800 [hep-th]].

[71] V. Cortes, P. Dempster, and T. Mohaupt. Time-like reductions of five-dimensional supergravity. *JHEP*. **1404** (2014) 190 [arXiv:1401.5672 [hep-th]].

[72] V. Cortes, C. Mayer, T. Mohaupt, and F. Saueressig. Special geometry of Euclidean supersymmetry. 1. Vector Multiplets. *JHEP*. **0403** (2004) 028 [hep-th/0312001].

[73] T. Mohaupt. Special geometry, black holes and Euclidean supersymmetry. [hep-th/0703037].

[74] J. B. Gutowski and W. A. Sabra. Euclidean N=2 Supergravity. *Phys. Lett. B*. **718** (2012) 610 [arXiv: 1209.2029[hep-th]].

[75] B. Chakrabarty and A. Virmani. Geroch Group Description of Black Holes. *JHEP*. **1411** (2014) 068 [arXiv:1408.0875 [hep-th]].

[76] R. Emparan and H. S. Reall. Generalized Weyl solutions. *Phys. Rev. D*. **65** 084025 (2002) [arXiv:hep-th/0110258].

[77] R. Emparan and H. S. Reall. A rotating black ring in five dimensions. *Phys. Rev. Lett.* **88** 101101 (2002) [hep-th/0110260].

[78] R. Emparan. Rotating circular strings, and infinite non-uniqueness of black rings. *JHEP*. **0403** (2004) 064 [hep-th/0308056].

[79] H. Elvang and P. Figueras. Black Saturn. *JHEP*. **0705** (2007) 050 [hep-th/0701035].

## Selbstständigkeitserklärung

Hiermit erkläre ich meine Dissertation selbstständig ohne fremde Hilfe verfasst zu haben und nur die angegebene Literatur und Hilfsmittel verwendet zu haben.

Despoina Katsimpouri
Potsdam, 04. März 2015